Biharmonic Problem with Steklov-type Boundary Conditions in a Half-Space

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Abstract The unique solvability of a biharmonic problem with Steklov-type boundary conditions in a half-space is studied under the assumption that the generalized solutions of this problem have a nite weighted Dirichlet integral. Depending on the value of the weight parameter, uniqueness theorems are proved or exact formulas are given for calculating the dimension of the solution space of a biharmonic problem with Steklov-type boundary conditions in a half-space.

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1. INTRODUCTION

For any integer $n \geq 2$, writing a typical point of $\mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$; we denote by $\mathbb{R}^n_+$ the "open half space" of $\mathbb{R}^n$:

$$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 1\}$$

and let

$$\partial\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = 1\} \equiv \mathbb{R}^{n-1}.$$

denote its boundary, and let $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ denote the Euclidean norm of x.

Let now $\Omega \equiv \mathbb{R}^n_+$ with the boundary $\partial\Omega \equiv \partial\mathbb{R}^n_+$. In Ω we consider the problem for the biharmonic equation

$$\Delta^2 u(x) = 0, \qquad x \in \Omega \tag{1}$$

with Steklov-type boundary conditions

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \left.\left(\frac{\partial \Delta u}{\partial \nu} + \tau u\right)\right|_{\partial\Omega} = 0, \tag{2}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer unit normal vector to $\partial\Omega$, $\tau \in C(\partial\Omega)$, $\tau \geq 0$, $\tau \not\equiv 0$, $\overline{\Omega} = \Omega \cup \partial\Omega$ is a closure of Ω.

Elliptic problems with parameters in the boundary conditions are called Steklov problems from their rst appearance in [30]. In the case of the biharmonic operator, these conditions were rst considered in [6], [18] and [29], who studied the isoperimetric properties of the rst eigenvalue.

Standard results on the regularity of elliptic problems are available in the monograph [11], which has already become a classic. Here, the authors consider both high-order linear and nonlinear boundary value problems, primarily with a biharmonic (polyharmonic) operator as the principal component. Using basic models, they explain in detail the importance and di erences of various boundary conditions. In studying linear problems, after a brief overview of existence theory, L^p, and Schauder estimates, the authors focus on positivity issues and on obtaining the necessary kernel estimates.

In [3] and [4], the authors study boundary value problems for an inhomogeneous biharmonic equation in a half-space $\mathbb{R}^n_+$, establishing the existence, uniqueness, and regularity of these problems in L^p -theory with $1 < p < \infty$. Note that the authors are interested in singular boundary conditions, and they consider data and present solutions that exist in weighted Sobolev spaces.

In [2], fundamental solutions to di erential operators lead to integral operators providing integral representation formulas for solutions to related di erential equations. Proper modi cations of the fundamental solutions result in integral operators which are related to certain boundary value problems. For complex partial di erential operators of arbitrary order in the plane, fundamental solutions are achievable by properly integrating the Cauchy kernel. Particular such complex model di erential operators are the polyanalytic and the polyharmonic operators. A hierarchy of integral operators is available for these model operators leading to polyanalytic Cauchy Schwarz and to polyharmonic Green, Neumann, Robin, and hybrid Green integral operators.

We also note paper [5], which explains for the biharmonic operator that the higher the order, the greater the variety of possible boundary value problems and associated hybrid Green's functions. The advantage of convoluted higher order Green's functions is that they allow to decompose the boundary value problem for a linear higher order Poisson equation into some for a system of rst order Poisson equations.

In [7], a weak solution of a mixed boundary value problem for a biharmonic equation in the plane is studied, in which, using the Green formula, the problem is transformed into a system of Fredholm integral equations for unknown data on di erent parts of the boundary. The existence and uniqueness of solutions of the system of boundary integral equations in the corresponding Sobolev spaces are also established.

In [10], the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwarz re ection principle in weighted L^q-space the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

In [13] and [14], the Green's function of the biharmonic Navier problem in the unit ball is studied. The author presents a representation of the Green's function in which the singularity of the fundamental solution of the biharmonic equation is explicitly expressed. Then, based on the Green's function, an integral representation of the solution to the Navier problem in the unit ball is presented.

For various classes of unbounded domains, the author in [19] [26] studied the properties of solutions to elliptic boundary value problems with a nite weighted Dirichlet (or energy) integral. In this paper, the niteness condition for the solution of the biharmonic problem is the Dirichlet integral with weight:

$$D_a(u,\Omega) \equiv \int_\Omega |x|^a \sum_{|\alpha|=2} |\partial^\alpha u|^2 \, dx < \infty,$$

where $a \in \mathbb{R}$ is a xed number and $\sqrt{\sum_{|\alpha|=2} |\partial^\alpha u|^2}$ denotes the Frobenius norm of the Hessian matrix of u.

Depending on n and a, the uniqueness of solutions to boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation is proven. In the case of non-uniqueness, the exact numbers of linearly independent solutions of boundary value problems, which are the dimensions of the spaces of solutions of these same boundary value problems, are found, and explicit formulas are provided.

Based on Hardy-type inequalities [9], [15] [17], in this paper we prove the uniqueness (or non-uniqueness) of solutions to the biharmonic problem with Steklov-type boundary conditions in a half-space.

Notation: $C_0^\infty(\Omega)$ is the space of in nitely di erentiable functions in Ω with compact support in Ω.

We denote by $H^m(\Omega, \Gamma)$, $\Gamma \subset \overline{\Omega}$, the Sobolev space of functions in Ω obtained by the completion of $C^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ with respect to the norm

$$||u; H^m(\Omega, \Gamma)|| = \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha u|^2 dx \right)^{1/2}, \quad m = 1, 2, \dots,$$

where $\partial^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_i \ge 0$ are integers, and $|\alpha| = \alpha_1 + \cdot \cdot \cdot + \alpha_n$; if $\Gamma = \emptyset$, we denote $H^m(\Omega, \Gamma)$ by $H^m(\Omega)$.

$\overset{\circ}{H}{}^m(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u; H^m(\Omega)||$;

$\overset{\circ}{H}{}^m_{loc}(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the family of semi-norms

$$\|u; H^m(\Omega \cap B_0(R))\| = \left(\int_{\Omega\cap B_0(R)} \sum_{|\alpha|\le m} |\partial^\alpha u|^2 \, dx \right)^{1/2}, \quad m = 1, 2, \dots,$$

for all open balls $B_0(R) := \{x : |x| < R\}$ in $\mathbb{R}^n$ for which $\Omega \cap B_0(R) \ne \emptyset$.

We set

$$D(u, \Omega) \equiv \int_\Omega \sum_{|\alpha|=2} |\partial^\alpha u|^2 \, dx, \quad D_a(u, \Omega) \equiv \int_\Omega |x|^a \sum_{|\alpha|=2} |\partial^\alpha u|^2 \, dx.$$

A cone K in $\mathbb{R}^n$ with vertex the origin is defined as a domain such that if $x \in K$, then $\lambda x \in K$ for all $\lambda > 0$.

2. DEFINITIONS & AUXILIARY STATEMENTS

Definition 1. A function u is a solution of the Steklov-type problem (1), (2), if $u \in H^2_{loc}(\Omega)$, $\partial u/\partial \nu = 0$ on $\partial\Omega$, such that for every function $\phi \in C_0^\infty(\mathbb{R}^n)$, $\partial\phi/\partial\nu = 0$ on $\partial\Omega$, the following integral identity holds

$$\int_\Omega \Delta u \, \Delta\phi \, dx - \int_{\partial\Omega} \tau \, u \, \phi \, ds = 0. \tag{3}$$

Note that the solutions of the Steklov-type problem are well defined for $\tau \in C(\partial\Omega)$. For $u \in H^4(\Omega)$ one may integrate by parts to find indeed that the solution of (3) satisfies to the Steklov-type problem (1),(2).

Lemma 1. Let $u(x)$ be a solution of equation (1) in Ω satisfying the boundary conditions (2) and the inequality $|u(x)| \le C(1 + |x|^k)$ for all $x \in \Omega$, where C is a positive constant and $k \ge 0$ is an integer. Then $u(x)$ is a polynomial of degree at most k, i. e., $u(x) = P(x)$ and $\operatorname{ord} P(x) \le k$.

Proof. This lemma is an analogue of Liouville's theorem for systems of equations. It is valid for general Douglis Nirenberg elliptic systems with constant coefficients. In the case when $\Omega \equiv \mathbb{R}^n$ it was proved in [8] .

The proof for $\Omega \equiv \mathbb{R}^n_+$ is similar. Namely, in the theory of elliptic systems the following Bernstein inequality is established:

$$\max_{|x|\le 1/2,\ |\alpha|=l} |\partial^\alpha u(x)| \le C_1 \max_{|x|\le 1} |u|, \quad l \ge 1, \tag{4}$$

where the constant C_1 is independent of u.

Let $u(x)$ be a solution of a homogeneous elliptic system with constant coefficients in the half-ball $|x| \le 1$, $x_n > 0$, and for $x_n = 0$ suppose that $u(x)$ satisfies the Shapiro Lopatinskii zero boundary condition [1] (for more details, see [28]). In (4) putting $x = \lambda y$ and using the homogeneity of the system and boundary conditions, we get

$$\max_{|x|\le 1/2,\ |\alpha|=l} |\partial^\alpha u(x)| \le C_1 \lambda^{-l} \max_{|x|\le\lambda} |u|.$$

Hence the hypotheses of Lemma 1 imply that

$$\max_{|x|\le 1,\ |\alpha|=l} |\partial^\alpha u(x)| \le C_2 \lambda^{k-l}.$$

Taking $l > k$ and letting λ tend to ∞, we see that $\partial^\alpha u(x) = 0$ for $|x| \le 1$. Hence, $u(x)$ is a polynomial of degree at most $l-1$ for $|x| \le 1$. Since the solution of an elliptic system is analytic, then $u(x)$ is a polynomial in $\mathbb{R}^n_+$. The condition $|u(x)| \le C(1+|x|^k)$ implies that the degree of this polynomial is at most k. The proof of the Lemma 1 is complete. □

Let us denote by $\mathrm{Ker}_a(\Delta^2)$ the class of functions that are solutions of the Steklov-type problem (1), (2), and satisfy the condition $D_a(u,\Omega) < \infty$, and by $\dim \mathrm{Ker}_a(\Delta^2)$ the dimension of the class $\mathrm{Ker}_a(\Delta^2)$.

3. MAIN RESULTS

Theorem 1. Let u be a solution of the Steklov-type problem (1), (2) in $\Omega \equiv \mathbb{R}^n_+$ with the condition $D(u,\Omega) < \infty$. Then $u \equiv 0$.

Proof. Let $2 \le n \le 4$. Let $u(x)$ be a solution to equation (1) in $\Omega \equiv \mathbb{R}^n_+$. We extend $u(x)$ to $\mathbb{R}^n$, setting $u(x) = 0$ in $\mathbb{R}^n \setminus \Omega$. Then $D(u,\mathbb{R}^n) < \infty$.

In the integral identity (3), substituting the function $\phi(x) = u(x)\vartheta_N(x)$, where $\vartheta_N(x) = \vartheta(\ln|x|/\ln N)$, $\vartheta \in C^\infty(\mathbb{R})$, $0 \le \vartheta \le 1$, $\vartheta(s) = 0$ as $s \ge 2$, $\vartheta(s) = 1$ as $s \le 1$, after elementary transformations, we get

$$\int_\Omega (\Delta u)^2 \vartheta_N(x)\,dx - \int_{\partial\Omega} \tau |u|^2 \vartheta_N(s)\,ds = J_1(u) + J_2(u), \tag{5}$$

where

$$J_1(u) = -2\int_\Omega \Delta u \nabla u \nabla \vartheta_N(x)\,dx, \quad J_2(u) = -\int_\Omega u\,\Delta u\,\Delta\vartheta_N(x)\,dx.$$

Let us show that $J_1(u) \to 0$ and $J_2(u) \to 0$ as $N \to \infty$. By the Cauchy Schwarz inequality we have

$$J_1(u) \equiv -2\int_\Omega \Delta u \nabla u \nabla\vartheta_N(x)\,dx \le 2C_2 \int_{\Omega\cap\{x:N<|x|<N^2\}} |\Delta u| \frac{|\nabla u|}{|x||\ln N|}\,dx \le C_3 J_3(u) J_4(u),$$

$$J_2(u) \equiv -\int_\Omega u\Delta u\Delta\vartheta_N(x)\,dx \le C_4 \int_\Omega |\Delta u| \frac{|u|}{|x|^2|\ln N|^2}\,dx \le C_4 J_3(u) J_5(u),$$

where

$$J_3(u) \equiv \left(\int_{|x|>N} |\Delta u|^2 dx\right)^{1/2}, \quad J_4(u) \equiv \left(\int_{N<|x|<N^2} \frac{|\nabla u|^2}{|x|^2|\ln N|^2} dx\right)^{1/2},$$

$$J_5(u) \equiv \left(\int_{N<|x|<N^2} \frac{|u|^2}{|x|^4 |\ln N|^4} dx \right)^{1/2}, \quad C_2, C_3, C_4 = \text{const}.$$

Since, by Hardy's inequality [15],

$$\chi(u) := \int_\Omega \left(\frac{|u|^2}{|x|^4 |\ln N|^4} + \frac{|\nabla u|^2}{|x|^2 |\ln N|^2} + |\nabla\nabla(u)|^2 \right) dx < \infty,$$

and $D(u, \Omega) < \infty$, then

$$J_3(u) \to 0, \quad J_4(u) \to 0, \quad J_5(u) \to 0 \quad \text{as} \quad N \to \infty.$$

This means that $J_1(u) \to 0$ and $J_2(u) \to 0$ as $N \to \infty$. Consequently, passing to the limit in (5) as $N \to \infty$, we get

$$\int_\Omega (\Delta u)^2 \vartheta_N(x)\, dx - \int_{\partial\Omega} \tau |u|^2 \vartheta_N(s)\, ds \to 0.$$

Using the integral identity

$$\int_\Omega (\Delta u)^2\, dx - \int_{\partial\Omega} \tau |u|^2 ds = 0,$$

we obtain that if u is the solution of the homogeneous Steklov-type problem (1),(2), then $u(x) = 0$ in Ω, i.e. $u(x) = Ax + B$ for $A, x \in \mathbb{R}^n$, where Ax denotes the standard scalar product of A and x.

The proof that from the given integral identity it follows that $\Delta u = 0$ and, consequently, $u(x) = Ax + B$, is based on the use of the properties of the energy functional and the conditions of the homogeneous Steklov-type problem when considering the zero eigenvalue ($\tau = 0$) in the domain $\Omega \equiv \mathbb{R}^n_+$.

Consequently, the only solutions of a homogeneous biharmonic Steklov-type problem in the half-space $\Omega \equiv \mathbb{R}^n_+$ with a nite weighted Dirichlet integral for $\tau = 0$ are linear functions of the form $u(x) = Ax + B$.

From the condition of boundedness of the Dirichlet integral $D(u, \Omega) < \infty$ it follows that the only admissible solution of the homogeneous Steklov-type problem in the half-space $\Omega \equiv \mathbb{R}^n_+$ is the trivial solution, that is $u(x) \equiv 0$ in Ω.

The relation

$$\int_{\partial\Omega} \tau |u|^2 ds = 0$$

implies that $u \equiv 0$ on a set of a positive measure on $\partial\Omega$.

Consider the case $n > 4$. Let u be a solution of equation (1) in $\Omega \equiv \mathbb{R}^n_+$. Extend u to $\mathbb{R}^n$ by setting $u = 0$ in $\mathbb{R}^n \setminus \Omega$. Then $D(u, \mathbb{R}^n) < \infty$.

By Hardy's inequality [17] there exists a constant C such that

$$\int_{\mathbb{R}^n} \left(\frac{|u - C|^2}{|x|^4} + \frac{|\nabla(u - C)|^2}{|x|^2} + |\nabla\nabla(u - C)|^2 \right) dx \leq C_5 D(u, \mathbb{R}^n) < \infty, \tag{6}$$

where $C_5 = \text{const}$.

Let $v = u - C$. We will show that

$$\int_{\mathbb{R}^n} |v|^{\frac{2n}{n-4}} dx < \infty. \tag{7}$$

For the unit ball $B_1(0) = \{x : |x| < 1\}$, by the embedding theorem we obtain [31]

$$\left(\int_{B_1(0)} |v|^{\frac{2n}{n-4}} dx\right)^{\frac{n-4}{2n}} \leq$$

$$C_6 \left[\left(\int_{B_1(0)} \frac{|v|^2}{|x|^4} dx\right)^{1/2} + \left(\int_{B_1(0)} \frac{|\nabla v|^2}{|x|^2} dx\right)^{1/2} + \left(\int_{B_1(0)} |\nabla\nabla v|^2 dx\right)^{1/2}\right], \tag{8}$$

where $C_6 = \text{const}$.

Let us make a change of variables $y = \lambda x$ in the integrals of inequality (8). We have

$$\left(\int_{|y|<\lambda} |v|^{\frac{2n}{n-4}} dy\right)^{\frac{n-4}{2n}} \leq$$

$$C_7 \left[\left(\int_{|y|<\lambda} \frac{|v|^2}{|y|^4} dy\right)^{1/2} + \left(\int_{|y|<\lambda} \frac{|\nabla v|^2}{|y|^2} dy\right)^{1/2} + \left(\int_{|y|<\lambda} |\nabla\nabla v|^2 dy\right)^{1/2}\right],$$

where the constant C_7 does not depend on λ and v. Due to condition (6) we obtain

$$\int_{|y|<\lambda} |v|^{\frac{2n}{n-4}} dy < M, \tag{9}$$

where the constant M does not depend on λ.

Since $v(x) = C$ on $\mathbb{R}^n \setminus \Omega$ and $\text{mes}_n(\mathbb{R}^n \setminus \Omega) = \infty$, it follows from inequality (9) that $C = 0$.

Let us show that $u(x) = 0,\ x \in \Omega$. Next, substituting into the integral identity (3) the function $\phi(x) = u(x)\vartheta_N(x)$, where $\vartheta_N(x) = \vartheta(|x|/N)$, $\vartheta \in C^\infty(\mathbb{R}^n)$, $0 \leq \vartheta \leq 1$, $\vartheta(s) = 0$ for $s \geq 2$, $\vartheta(s) = 1$ for $s \leq 1$, as in the proof of (5), we nd

$$\int_\Omega (\Delta u)^2\, \vartheta_N(x)\, dx - \int_{\partial\Omega} \tau\, |u|^2 \vartheta_N(s)\, ds = J_1(u) + J_2(u),$$

As above, it is easy to notice that $J_1(u) \to 0$ and $J_2(u) \to 0$ for $N \to \infty$. Hence,

$$\int_\Omega (\Delta u)^2\, \vartheta_N(x) dx - \int_{\partial\Omega} \tau\, |u|^2 \vartheta_N(s)\, ds \to 0 \quad \text{as} \quad N \to \infty.$$

Using the integral identity

$$\int_\Omega (\Delta u)^2\, dx - \int_{\partial\Omega} \tau\, |u|^2 ds = 0,$$

we obtain that if u is the solution of the Steklov-type problem (1), (2), then $u(x) = 0$ in Ω, i.e. $u(x) = Ax + B$ for $A, x \in \mathbb{R}^n$, where Ax denotes the standard scalar product of A and x.

The proof that from the given integral identity it follows that $\Delta u = 0$ and, consequently, $u(x) = Ax + B$, is based on the use of the properties of the energy functional and the conditions of the homogeneous Steklov-type problem when considering the zero eigenvalue ($\tau = 0$) in the domain $\Omega \equiv \mathbb{R}^n_+$.

Consequently, the only solutions of a homogeneous biharmonic Steklov-type problem in the half-space $\Omega \equiv \mathbb{R}^n_+$ with a nite weighted Dirichlet integral for $\tau = 0$ are linear functions of the form $u(x) = Ax + B$.

From the condition of boundedness of the Dirichlet integral $D(u, \Omega) < \infty$ it follows that the only admissible solution of the homogeneous Steklov-type problem in the half-space $\Omega \equiv \mathbb{R}^n_+$ is the trivial solution, that is $u(x) \equiv 0$ in Ω.

The relation

$$\int_{\partial\Omega} \tau\, |u|^2 ds = 0$$

implies that $u \equiv 0$ on a set of a positive measure on $\partial\Omega$.

The proof of the theorem is complete.

□

Theorem 2. The Steklov-type problem (1),(2) with the condition $D_a(u, \Omega) < \infty$ has only the trivial solution for $-n \leq a < \infty$.

Proof. First, let us consider the case when $a \geq 0$, $n \geq 2$. Indeed, if $a \geq 0$, then $\mathrm{Ker}_a(\Delta^2) \subset \mathrm{Ker}(\Delta^2)$ and

$$\dim \mathrm{Ker}_a(\Delta^2) \leq \dim \mathrm{Ker}(\Delta^2).$$

Taking into account Theorem 1 it follows that

$$\dim \mathrm{Ker}_a(\Delta^2) = 0.$$

Let us now consider the case when $-n \leq a < 0$, $n \geq 2$. Let

$$u(x) \in \mathrm{Ker}_a(\Delta^2), \qquad \text{where} \quad -n \leq a < 0.$$

According to Lemma 1, the solution $u(x)$ has the form

$$u(x) = P(x),$$

where $P(x)$ is a polynomial, $\mathrm{ord}\, P(x) \leq k$.

Let us prove that $\mathrm{ord}\, P(x) \leq 1$. Let ord $P(x) = k$ and $k \geq 2$. Then inside some cone K the following inequality holds:

$$|\partial^\alpha P(x)| \geq C|x|^{k-2}.$$

Hence,

$$\infty > D_a(u, \Omega) = D_a(P(x), \Omega) = \int_\Omega |x|^a \sum_{|\alpha|=2} |\partial^\alpha P(x)|^2\, dx$$

$$= \int_\Omega |x|^a |\partial^\alpha P(x)|^2\, dx \geq C \int_{K \cap \{|x|>N\}} |x|^a |x|^{2(k-2)}\, dx.$$

The obtained integral converges only if $a + 2k - 4 + n < 0$, i.e. $k < 2$, so $\mathrm{ord}\, P(x) = 1$.

Therefore, $u(x) = P(x)$ and $\mathrm{ord}\, P(x) = 1$. It is easy to see that

$$D(u, \Omega) = D(P(x), \Omega) = 0,$$

i.e. $u(x) \in \mathrm{Ker}(\Delta^2)$. Hence,

$$\mathrm{Ker}_a(\Delta^2) \subset \mathrm{Ker}(\Delta^2) \qquad \text{for} \quad -n \leq a.$$

On the other hand, it is obvious that

$$\mathrm{Ker}(\Delta^2) \subset \mathrm{Ker}_a(\Delta^2) \qquad \text{for} \quad a < 0.$$

So then

$$\mathrm{Ker}_a(\Delta^2) = \mathrm{Ker}(\Delta^2)$$

and

$$\dim \mathrm{Ker}_a(\Delta^2) = \dim \mathrm{Ker}(\Delta^2).$$

By virtue of Theorem 1, we have $\dim \mathrm{Ker}(\Delta^2) = 0$. Hence,

$$\dim \mathrm{Ker}_a(\Delta^2) = 0.$$

The theorem is proved. □

Theorem 3. The Steklov-type problem (1),(2) with the condition $D_a(u, \Omega) < \infty$ has $k(r, n)$ linearly independent solutions for $-2r - n + 2 \leq a < -2r - n + 4$, $r \geq 2$, where

$$k(r, n) = \binom{r+n}{n} - \binom{r+n-4}{n} - \binom{r+n-2}{n-1} - \binom{r+n-4}{n-1};$$

here $\binom{n}{k}$ is the (n, k) binomial coe cient, $\binom{n}{k} = 0$ for $k > n$.

Proof. Let u is a solution of the Steklov-type problem (1), (2) and $D_a(u, \Omega) < \infty$.

Then, according to Lemma 1, the solution $u(x)$ is a polynomial of order no higher than k, i.e. $u(x) = P(x)$, $\mathrm{ord}\, P(x) \leq k$.

Let us show that $k = r$. Let $\mathrm{ord}\, P(x) = s > r$. Then inside some cone K the following inequality holds:

$$|\partial^\alpha P(x)| \geq C|x|^{s-2}.$$

Hence,

$$\infty > D_a(u, \Omega) = D_a(P(x), \Omega) = \int_\Omega |x|^a \sum_{|\alpha|=2} |\partial^\alpha P(x)|^2 dx$$

$$= \int_\Omega |x|^a |\partial^\alpha P(x)|^2 dx \geq C \int_{K \cap \{|x|>N\}} |x|^a |x|^{2(s-2)} dx.$$

The resulting integral converges only if $a + 2s - 4 + n < 0$, i.e.

$$s < \frac{(4 - a - n)}{2} \leq r + 1.$$

Since s is an integer, then $s \leq r$. Thus, $k = r$ and $u(x) = P(x)$, $\mathrm{ord}\, P(x) \leq r$.

The dimension of all polynomials in $\mathbb{R}^n$ of degree not higher than r is equal to $\binom{r+n}{n}$ [27]. Then the dimension of all biharmonic polynomials in $\mathbb{R}^n$ of degree not higher than r is equal to

$$\binom{r+n}{n} - \binom{r+n-4}{n},$$

since the biharmonic equation represents the equality to zero of some polynomial of degree $(r - 4)$ in $\mathbb{R}^n$.

The conditions for the vanishing of a polynomial and its derivatives with respect to x_n on the hyperplane $x_n = 1$ represent the equality to zero of some polynomials in $(n - 1)$ variables of degree $r - 1$ and $r - 3$, respectively.

Consequently, the dimension of the space of solutions of the Steklov-type problem (1), (2) is equal to

$$k(r, n) = \binom{r+n}{n} - \binom{r+n-4}{n} - \binom{r+n-2}{n-1} - \binom{r+n-4}{n-1}.$$

The theorem is proved.

□

4. DISCUSSION

The study of elliptic boundary value problems in domains with both compact and non-compact boundaries is a very relevant area of research in mathematical physics.

The next development for such problems will be to consider boundary value problems for elliptic equations and systems in domains with non-smooth boundaries, under the condition that the weighted Dirichlet (or energy) integral is bounded at in nity.

As is known, elliptic problems with parameters in the boundary conditions were rst considered in the scienti c works of Steklov, and therefore, for both the Laplace equation and the biharmonic equation, such problems are known as Steklov or Steklov-type problems.

The biharmonic problem with Steklov-type boundary conditions at $\tau = 0$ becomes the Farwig problem [25], which was rst generalized for the polyharmonic equation in [26].

Biharmonic problems in the presence of parameter τ in the boundary conditions are of a general nature and nd practical application in engineering, medicine and other elds.

5. CONCLUSIONS

In conclusion, it is worth adding that in the future, elliptic problems with boundary conditions both with and without a parameter will be considered, but in domains with a non-smooth boundary.

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7. CONFLICT OF INTEREST

The author of this paper declares that he has no con icts of interest.

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