A Fixed Point Result for Generalized Expansion onto Self-Mappings on Complete CM-Space

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Abstract

In this paper, we obtain a fixed point result for generalized expansion onto self- mappings on Complete CM-Spaces .This result is an extension and improved result of the some of the existing results in this literature.

Key Words: Cone metric space, continuous mapping fixed point.

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1. Introduction

Cone metric space was introduced by the mathematicians Huang and Zhang [7] . They obtained some fixed point results in cone metric space. Later on many authors were inspired by these results, they have been extending these results in different dimensions (see for e.g. [1-15]). In recent developments in this area for non explosive map in cone metric spaces (see for e.g. [1-6], [8-11]). Recently, Aaage and Salunke [6] proved some fixed point theorems for expansion onto mappings on cone metric spaces . In this paper, we obtained affixed point theorem for expansion mappings on cone metric spaces.

2. Preliminaries

The following ar useful in our main results which are due to [7].

Definition 2.1. Let B be a real Banach space and Q is a subset of B, Q is called a cone iff

(i) Q is closed , non empty and $Q \neq \{0\}$,
(ii) $\alpha x + \beta y \in Q$ for all $x, y \in Q$ and α, β are non negative numbers,
(iii) $Q \cap (-Q) = \{0\}$.

Definition 2.2. Given a cone $Q \subset B$, we define a partial ordering $\leq$ on B with respect to Q by

$x << y$ if $x < y$ and $x \neq y$. we shall write $x << y$ if $y-x \in$ interior of Q . The cone Q is called normal if there is a number $M > 0$ such that for all $x, y \in B$,

$$0 \leq x \leq y \text{ implies } \| x \| \leq M \|y\|.$$

The least positive number M satisfying the above is called the normal constant of Q.

Definition 2.3. Let X be a non- empty set . Suppose the $\rho : X \times X \to X$ satisfies the following

(a) $\rho(x, y) \geq 0$ for all $x, y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;
(b) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;
(c) $\rho(x, y) \leq \rho(x, y) + (y, z)$, for all $x, y ,z \in X$.
Then ρ is called a cone metric on X and (X, ρ) is called a cone metric space.

Definition 2.4. Let (X, ρ) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $\varepsilon > 0$ there is an $n_0 \in N$ such that $\rho(x_n , x) < \varepsilon$,for all $n > n_0$, then $\{x_n\}$ is said to be convergence , $\{x_n\}$ convergences to x and x is the limit point of $\{x_n\}$. We denote this by $\log_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to \infty$.

Definition 2.5. Let (X, ρ) be a cone metric space .Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $\varepsilon > 0$ there is an $n_0 \in N$ such that $\rho(x_n , x_{n+m}) < \varepsilon$,for all $n > n_0$, then $\{x_n\}$ is called a Cauchy sequence in X, $\{x_n\}$ convergences to x and x is the limit point of $\{x_n\}$.

Definition 2.6. Let (X, ρ) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

3. Main Results

Theorem. 3.1.Let (X, ρ) be a CCM-Space and the mapping $A: X \to X$ is continuous onto satisfies the following:

$$\rho(Ax, Ay) \geq \alpha[\rho(x, Ax) + \rho(y, Ay)] + \beta[\rho(x, Ay) + \rho(y, Ax)],$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $½ < \alpha + \beta \leq 1$. Then A has a fixed point.

Proof: For each $x_0 \in X$. Sine A is onto there exists $x_1 \in X$ such that $x_0 = Ax_1$ similarly for each $n \geq 1$ there exists $x_{n+1} \in X$ such that $x_n = Ax_{n+1}$. If $x_{n-1} = x_n$ then x_n is a fixed point of A . Thus we suppose that $x_{n-1} \neq x_n$ for all $n \geq 1$. Then

$$\rho(x_n , x_{n-1}) = \rho(Ax_{n+1} , Ax_n)$$

$$\geq \alpha[\rho(x_{n+1}, Ax_{n+1}) + \rho(x_n, Ax_n)] + \beta[\rho(x_{n+1}, Ax_n) + \rho(x_n, Ax_{n+1})],$$

$\geq \alpha[\ \rho(x_{n+1,} x_n) + \rho(x_{n,} x_{n-1})] + \beta[\ \rho(x_{n+1,} x_{n-1}) + \rho(x_{n,} x_n)],$

$= (\alpha + \beta)\ \rho(x_{n+1,} x_n) + (\alpha+ \beta)\ \rho(x_{n,} x_{n-1})\ ,$

$\rho(x_n\ , x_{n-1}\ ) \ \leq 1\text{-}\ (\ \alpha + \beta)/(\ \alpha+\beta)\rho(x_{n-1\ ,} x_n)\ ,$

$\leq\ k\ \rho(x_{n-1\ ,} x_n)$, where , $k = 1\text{-}\ (\ \alpha + \beta) \setminus (\ \alpha+\beta\ ) < 1.$

From then we get that

$\rho(x_n\ , x_{n-1}\ ) \ \leq\ k^n\ \rho(x_{0\ ,} x_1)$, where , $k =\ 1\text{-}\ (\ \alpha + \beta) \setminus (\ \alpha+\beta\ )$ and $0 \leq\ k < 1.$

Now for $n > m$, $\rho(x_n\ , x_{n-1}\ )$ we have

$\rho(x_n\ , x_m\ ) \ \leq \rho(x_n\ , x_{n+1}\ ) \ \ + \rho(x_{n+1}\ , x_{n+2}\ ) \ \ + \ldots + \rho(x_{m-1}\ , x_m\ ) \ \ ,$

$\leq\ (k^n + k^{n+1} + \ldots + k^{m-1})\ \rho(x_{0\ ,} x_1),$

$\leq k^n/1\text{-}k\ \ \rho(x_{0\ ,} x_1).$

Let $0 \leq e$ be given. Choose a natural number N_1 such that $k^n/1\text{-}k\ \ \rho(x_{0\ ,} x_1) \leq e$ for all $n \geq N_1$.

Thus , $\rho(x_n\ , x_m\ ) \ \leq e$ for $n > m$. Therefore, $\{x_n\}$, $n \geq 1$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is a CCM-space there exists $x^* \epsilon X$ such that $x_n \to x^*$ as $n \to \infty$. If A is continuous then

$\rho(Ax^*, x^*) \leq \rho(Ax_n, Ax^*) + \rho(Ax^*\ , x^*) \to 0$, as $n \to \infty$. Since $x_n \to x^*$ and $Ax_n \to A\ x^*$,

as $n \to \infty$. Therefore, $\rho(Ax^*\ , x^*) = 0$ and so $Ax^* = x^*$. Then A has a fixed point in X. This completes the proof of the theorem.

Remark: If we take $\beta = 0$ in the above theorem we get the Theorem2.3. of [6].

Conclusion: Our results are more general than the results of [6].

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