

SUBCLASS OF ANALYTIC FUNCTIONS CONNECTED WITH HURWITZ-LERCH ZETA FUNCTION

J.R.WADKAR¹, GANESH B. DAPKE², AND P.THIRUPATHI REDDY³

¹ Department of Mathematics, Late Baburao Patil Arts and Science College,
Dist. Hingoli, M.S, India.

² Department of Mathematics, Sharad Institute of Technology, College of Engineering,
Yarav-lchalkaranji Dist.Kolhapur, Maharashtra, India.

³ Department of Mathematics, DRK Institute of Science and Technology,
Bowarmpet-500 043, Hyderabad, Telangana, India.

ABSTRACT. the Hurwitz zeta function plays a crucial role in mathematics and physics, offering insights into a wide range of phenomena and serving as a bridge between different areas of study. Making use of the Hurwitz-Lerch zeta operator, we introduce a new subclass of analytic functions defined in the open unit disk and investigate its various characteristics. Further, we obtain some usual properties of the geometric function theory such as coefficient bounds, extreme points, closure theorems, radius of starlikeness, convexity, partial sums and neighborhood results.

2020 Mathematics Subject Classification: 30C45.

Keywords and Phrases: analytic, starlike, convexity, partial sums, neighborhood.

1. INTRODUCTION

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

A function $u \in A$ is a starlike function of the order ξ , $0 \leq \xi < 1$, if it satisfies

$$\Re \left\{ \frac{zu'(z)}{u(z)} \right\} > \xi, \quad z \in U. \quad (1.2)$$

We denote this class with $S^*(\xi)$.

A function $u \in A$ is a convex function of the order ξ , $0 \leq \xi < 1$, if it fulfils

$$\Re \left\{ 1 + \frac{zu''(z)}{u'(z)} \right\} > \xi, \quad z \in U. \quad (1.3)$$

We denote this class with $K(\xi)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in U respectively. For $f \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (1.4)$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z), \quad (z \in U). \quad (1.5)$$

Note that $u * g \in A$.

Let T denotes the class of functions analytic in U that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \quad (z \in U) \tag{1.6}$$

and let $T^*(\xi) = T \cap S^*(\xi)$, $C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [17].

The zeta function has various properties and connections to number theory and other areas of mathematics. It has been studied extensively in the context of special functions and analytic number theory. The zeta function finds applications in various areas of mathematics and physics. Some of its applications include: Number Theory, Quantum Field Theory, String Theory, Analytic Number Theory and Algebraic Geometry. These are just a few examples of the applications of the zeta function. Its versatility and importance in various branches of mathematics and physics make it a subject of ongoing research and interest among mathematicians and physicists. The study of operators is essential in geometric function theory and its associated topics. In recent years, there has been a surge in interest in issues involving evaluations of various families of series associated with the Riemann and Hurwitz zeta functions, as well as their extensions and generalisations such as the Hurwitz-Lerch zeta function. These functions ascend naturally in many branches of analytic function theory and their studies have plentiful important applications in mathematics [1]. As an overview of both Riemann and Hurwitz zeta functions, the so-called Hurwitz-Lerch zeta function is defined in [7]. Hurwitz- Lerch Zeta function $\Phi(z, s, a)$ defined in [20] given by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.7}$$

($a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}; \Re(s) > 1$) and $|z| < 1$ where, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$, ($\mathbb{Z} := \{\pm 0, \pm 1, \pm 2, \pm 3, \dots\}$). It is clear that Φ is an analytic function in both variables s and z in a suitable region and it reduces to the ordinary Lerch zeta function when $z = e^{2\pi i \lambda}$. Moreover, Φ yields the following known result [7].

$$\Phi(z, 1, a) = a^{-1} {}_2F_1(a, 1; a + 1, z),$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Several interesting properties and characteristics of the Hurwitz- Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [6], and (also see [11]) the reference stated therein. The double zeta function of Barnes [3] is defined by

$$\zeta(x, a, \sigma) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (m + a + n\sigma)^{-x},$$

where $a \neq 0$ and σ is a non zero complex number with $|\arg(\sigma)| < \pi$. Bin- Saad [5] posed a generalized double zeta function of the form

$$\zeta_{\sigma}^{\nu}(z, s, a) = \sum_{n=0}^{\infty} (\nu)_n \Phi(z, s, a + n\sigma) \frac{z^n}{n!}$$

where $\sigma \in \mathbb{C} \setminus \{0\}; \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; a \in \mathbb{C} \setminus \{-(m + \sigma n)\}$, $n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $|s| < 1; |z| < 1$ and Φ is the Hurwitz-Lerch zeta function distinct by (1.7) and $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \begin{cases} 1, & n = 0 \\ \nu(\nu + 1)(\nu + 2)\dots(\nu + n - 1), & n \in \mathbb{N} \end{cases} \tag{1.8}$$

In [15], Rabhaw and Darus defined a function as follows:

$$\Theta_n(z, s, a) = \frac{\Phi(z, s, a + n\sigma)}{\Phi(z, s, a)}, \quad n \in \mathbb{N}_0 \tag{1.9}$$

It is clear that $\Theta_0(z, s, a) = 1$. Now consider the function

$$\Upsilon_{\nu}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \Theta_n(z, s, a) z^n, \tag{1.10}$$

which implies

$$z\Upsilon_\nu(z, s, a) = z + \sum_{n=2}^{\infty} \frac{(\nu)_{n-1}}{(n-1)!} \Theta_{n-1}(z, s, a) z^n$$

Thus,

$$z\Upsilon_\nu(z, s, a) * (z\Upsilon_\nu(z, s, a))^{-1} = \frac{z}{(1-z)^\delta} = z + \sum_{n=2}^{\infty} \frac{(\delta)_{n-1}}{(n-1)!} z^n, \quad \delta > -1$$

poses a linear operator

$$\mathfrak{J}_\nu^\delta(z, s, a)u(z) = (z\Upsilon_\nu(z, s, a))^{-1} * u(z) = z + \sum_{n=2}^{\infty} \frac{(\delta)_{n-1}}{(\nu)_{n-1} \Theta_{n-1}(z, s, a)} a_n z^n \tag{1.11}$$

where $\sigma \in \mathbb{C} \setminus \{0\}$; $\nu \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $a \in \mathbb{C} \setminus \{-(m + \sigma n)\}$, $n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $|s| < 1$; $|z| < 1$ and $\Theta_n(z, s, a)$ is defined in (1.9). It is clear that

$$\mathfrak{J}_\nu^\delta u(z) = \mathfrak{J}_\nu^\delta(z, s, a)u(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \tag{1.12}$$

where

$$\Psi_n = \frac{(\delta)_{n-1}}{(\nu)_{n-1} \Theta_{n-1}(z, s, a)}$$

Now, by making use of the Hurwitz - Lerch zeta operator, we define a new subclass of functions motivated by the recent work of Thirupathi Reddy and Venkateswarlu [23], Venkateswarlu et al [24,25] and Niranjana et al [13].

Definition 1.1. For $-1 \leq v < 1$ and $0 \leq \tau < 1$, we let $S(\tau, v)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re \left\{ \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - v \right\} \geq \left| \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - 1 \right|, \tag{1.13}$$

for $z \in U$.

The aims of the double zeta function encompass a wide range of mathematical and physical phenomena, making it a subject of significant interest and investigation in both pure and applied mathematics. The aim of this paper is to introduce a new subclass of analytic functions defined in the open unit disk and investigate its various characteristics. Further, we obtain some usual properties of the geometric function theory such as coefficient bounds, extreme points, closure theorems, radius of starlikeness, convexity, partial sums and neighborhood results.

2. COEFFICIENT BOUNDS

In this section we obtain a necessary and sufficient condition for function $u(z)$ is in the classes $S(\tau, v)$ and $TS(\tau, v)$.

Theorem 2.1. The function u defined by (1.1) is in the class $S(\tau, v)$ if

$$\sum_{n=2}^{\infty} [1 + \tau(n-1)][2n - (v+1)] \Psi_n |a_n| \leq 1 - v, \tag{2.1}$$

where $-1 \leq v < 1$, $0 \leq \tau \leq 1$.

Proof. It suffices to show that

$$\left| \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - 1 \right| - \Re \left\{ \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - 1 \right\} \leq 1 - v.$$

we have

$$\begin{aligned} & \left| \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - 1 \right| - \Re \left\{ \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - 1 \right\} \\ & \leq 2 \left| \frac{z(\mathfrak{J}_\nu^\delta u(z))' + \tau z^2(\mathfrak{J}_\nu^\delta u(z))''}{(1-\tau)\mathfrak{J}_\nu^\delta u(z) + \tau z(\mathfrak{J}_\nu^\delta u(z))'} - 1 \right| \\ & \leq \frac{2 \sum_{n=2}^{\infty} (n-1)[1 + \tau(n-1)]\Psi_n |a_n|}{1 - \sum_{n=2}^{\infty} [1 + \tau(n-1)]\Psi_n |a_n|}. \end{aligned}$$

This last expression is bounded above by $(1 - v)$ by

$$\sum_{n=2}^{\infty} [1 + \tau(n-1)][2n - (v+1)]\Psi_n |a_n| \leq 1 - v,$$

and hence the proof is complete.

Theorem 2.2. *A necessary and sufficient condition for $u(z)$ of the form (1.6) to be in the class $TS(\tau, v)$, $-1 \leq v < 1, 0 \leq \tau \leq 1$, is that*

$$\sum_{n=2}^{\infty} [1 + \tau(n-1)][2n - (v+1)]\Psi_n |a_n| \leq 1 - v. \tag{2.2}$$

Proof.

□

In view of Theorem 2.1, we need only to prove the necessity.

If $u \in TS(\tau, v)$ and z is real then

$$\frac{1 - \sum_{n=2}^{\infty} n[1 + \tau(n-1)]\Psi_n a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [1 + \tau(n-1)]\Psi_n a_n z^{n-1}} - v \geq \left| \frac{\sum_{n=2}^{\infty} (n-1)[1 + \tau(n-1)]\Psi_n |a_n|}{1 - \sum_{n=2}^{\infty} [1 + \tau(n-1)]\Psi_n |a_n|} \right|.$$

Letting $z \rightarrow 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty} [1 + \tau(n-1)][2n - (v+1)]\Psi_n |a_n| \leq 1 - v.$$

□

Theorem 2.3. *The class $TS(\tau, v)$ is convex set.*

Proof. Let the function

$$u_j = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1, 2, \tag{2.3}$$

be in the class $TS(\tau, v)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \zeta u_1(z) + (1 - \zeta)u_2(z), \quad 0 \leq \zeta \leq 1,$$

is in the class $TS(\tau, v)$ then

$$h(z) = z - \sum_{n=2}^{\infty} [\zeta a_{n,1} + (1 - \zeta)a_{n,2}]z^n.$$

By simple computation with the aid of Theorem 2.2 gives,

$$\begin{aligned} & \sum_{n=2}^{\infty} [1 + \tau(n - 1)][2n - (v + 1)]\Psi_n \zeta a_{n,1} \\ & + \sum_{n=2}^{\infty} [1 + \tau(n - 1)][2n - (v + 1)]\Psi_n (1 - \zeta) a_{n,2} \\ & \leq \zeta(1 - v) + (1 - \zeta)(1 - v) \\ & \leq 1 - v, \end{aligned}$$

which implies that $g \in TS(\tau, v)$.

Hence $TS(\tau, v)$ is convex. □

3. EXTREME POINTS

The proof of Theorem 3.1, follows on lines similar to the proof of the theorem on extreme points given in Silverman [23].

Theorem 3.1. *Let $u_1(z) = z$ and*

$$u_n(z) = z - \frac{1 - v}{[1 + \tau(n - 1)][2n - (v + 1)]\Psi_n} z^n, \tag{3.1}$$

for $n = 2, 3, \dots$. Then $u(z) \in TS(\tau, v)$ if and only if $u(z)$ can be expressed in the form $u(z) = \sum_{n=2}^{\infty} \zeta_n u_n(z)$, where $\zeta_n \geq 0$ and $\sum_{n=1}^{\infty} \zeta_n = 1$.

Next we prove the following closure theorem.

4. CLOSURE THEOREM

Theorem 4.1. *Let the function $u_j(z), j = 1, 2, \dots, l$ defined by (2.3) be in the classes $TS(\tau, v_j), j = 1, 2, \dots, l$ respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{l} \sum_{n=2}^{\infty} \left(\sum_{j=1}^l a_{n,j} \right) z^n$$

is in the class $TS(\tau, v)$, where $v = \min_{1 \leq j \leq l} \{v_j\}, -1 \leq v_j \leq 1$.

Proof. Since $u_j(z) \in TS(\tau, v_j, \varrho), j = 1, 2, \dots, l$ by applying Theorem 2.2 to (2.3), we observe that

$$\begin{aligned} & \sum_{n=2}^{\infty} [1 + \tau(n - 1)][2n - (v + 1)]\Psi_n \left(\frac{1}{l} \sum_{j=1}^l a_{n,j} \right) \\ & = \frac{1}{l} \sum_{j=1}^l \left(\sum_{n=2}^{\infty} [1 + \tau(n - 1)][2n - (v + 1)]\Psi_n a_{n,j} \right) \\ & \leq \frac{1}{l} \sum_{j=1}^l (1 - v_j) \\ & \leq 1 - v \end{aligned}$$

which in view of Theorem 2.2, again implies that $h(z) \in TS(\tau, v)$ and so the proof is complete. □

Theorem 4.2. *Let $u \in TS(\tau, v)$. Then*

- (1). u is starlike of order $\omega, 0 \leq \omega < 1$, in the disc $|z| < r_1$

i.e., $\Re \left\{ \frac{zu'(z)}{u(z)} \right\} > \omega, |z| < r_1$, where

$$r_1 = \inf_{n \geq 2} \left\{ \left(\frac{1 - \omega}{n - \omega} \right) \frac{[1 + \tau(n - 1)][2n - (v + 1)]\Psi_n}{1 - v} \right\}^{\frac{1}{n-1}}.$$

- (2). u is convex of order ω , $0 \leq \omega < 1$, in the disc $|z| < r_1$
 i.e., $\Re \left\{ 1 + \frac{zu''(z)}{u'(z)} \right\} > \omega$, $|z| < r_2$, where

$$r_2 = \inf_{n \geq 2} \left\{ \left(\frac{1 - \omega}{n - \omega} \right) \frac{[1 + \tau(n - 1)][2n - (v + 1)]\Psi_n}{1 - v} \right\}^{\frac{1}{n}}.$$

Each of these results are sharp for the extremal function $u(z)$ given by (3.1).

Proof. Given $u \in A$ and u is starlike of order ω , we have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| < 1 - \omega. \tag{4.1}$$

For the left hand side (4.1), we have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n - 1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

The last expression is less than $1 - \omega$ if

$$\sum_{n=2}^{\infty} \frac{n - \omega}{1 - \omega} a_n |z|^{n-1} < 1.$$

Using the fact, that $u \in TS(\tau, v)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[1 + \tau(n - 1)][2n - (v + 1)]\Psi_n}{1 - v} a_n < 1.$$

We can say (4.1) is true if

$$\frac{n - \omega}{1 - \omega} |z|^{n-1} < \frac{[1 + \tau(n - 1)][2n - (v + 1)]\Psi_n}{1 - v}$$

Or equivalently,

$$|z|^{n-1} < \frac{(1 - \omega)[1 + \tau(n - 1)][2n - (v + 1)]\Psi_n}{(n - \omega)(1 - v)}$$

which yields the starlikeness of the family.

(2). Using the fact that u is convex if and only if zu' is starlike, we can prove (2), on lines similar to the proof of (1). □

5. PARTIAL SUMS

Following the earlier works by Silverman [18] and Silvia [19] on partial sums of analytic functions. We consider in this section partial sums of functions in this class $S(\tau, v)$ and obtain sharp lower bounds for the ratios of real part of $u(z)$ to $u_q(z)$ and $u'(z)$ to $u'_q(z)$.

Theorem 5.1. Let $u(z) \in S(\tau, v)$. Define the partial sums $u_1(z)$ and $u_q(z)$ by

$$u_1(z) = z \text{ and } u_q(z) = z + \sum_{n=2}^q a_n z^n, \quad (q \in \mathbb{N} \setminus \{1\}). \quad (5.1)$$

Suppose that $\sum_{n=2}^{\infty} d_n |a_n| \leq 1$,

$$\text{where } d_n = \frac{[1 + \tau(n-1)][2n - (v+1)]\Psi_n}{1-v} \quad (5.2)$$

Then $u \in S(\tau, v)$.

$$\text{Further more, } \Re \left[\frac{u(z)}{u_q(z)} \right] > 1 - \frac{1}{d_{q+1}}, \quad (z \in E, \quad q \in \mathbb{N} \setminus \{1\}) \quad (5.3)$$

$$\text{and } \Re \left[\frac{u_q(z)}{u(z)} \right] > \frac{d_{q+1}}{1 + d_{q+1}}. \quad (5.4)$$

Proof. For the coefficients d_n given by (5.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \quad (5.5)$$

$$\text{Therefore we have } \sum_{n=2}^q |a_n| + d_{q+1} \sum_{n=q+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1 \quad (5.6)$$

by using the hypothesis (5.2). By setting

$$\begin{aligned} g_1(z) &= d_{q+1} \left[\frac{u(z)}{u_q(z)} - \left(1 - \frac{1}{d_{q+1}} \right) \right] \\ &= 1 + \frac{d_{q+1} \sum_{n=q+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^q a_n z^{n-1}} \end{aligned} \quad (5.7)$$

and applying (5.6), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_{q+1} \sum_{n=q+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^q |a_n| - d_{q+1} \sum_{n=q+1}^{\infty} |a_n|} \leq 1 \quad (5.8)$$

which readily yields the assertion (5.3) of Theorem 5.1. In order to see that

$$u(z) = z + \frac{z^{q+1}}{d_{q+1}} \text{ gives sharp result, we observe that for } z = re^{\frac{i\pi}{q}} \text{ that} \quad (5.9)$$

$$\frac{u(z)}{u_q(z)} = 1 + \frac{z^q}{d_{q+1}} \rightarrow 1 - \frac{1}{d_{q+1}} \text{ as } z \rightarrow 1^-.$$

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + d_{q+1}) \left(\frac{u_q(z)}{u(z)} - \frac{d_{q+1}}{1 + d_{q+1}} \right) \\ &= 1 - \frac{(1 + d_{q+1}) \sum_{n=q+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \quad (5.10)$$

and making use of (5.6), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{q+1}) \sum_{n=q+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^q |a_n| - (1 - d_{q+1}) \sum_{n=q+1}^{\infty} |a_n|}$$

which leads is immediately to the assertion (5.4) of Theorem 5.1.

The bound in (5.4) is sharp for each $q \in \mathbb{N}$ with the external function $u(z)$ given by (5.9). The proof of the Theorem 5.1 is thus complete. \square

Theorem 5.2. *If $u(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$\Re \left[\frac{u'(z)}{u'_q(z)} \right] \geq 1 - \frac{q + 1}{d_{q+1}}. \tag{5.11}$$

Proof. By setting

$$\begin{aligned} g(z) &= d_{q+1} \left[\frac{u'(z)}{u'_q(z)} \right] - \left(1 - \frac{q + 1}{d_{q+1}} \right) \\ &= \frac{1 + \frac{d_{q+1}}{q+1} \sum_{n=q+1}^{\infty} na_n z^{n-1} + \sum_{n=2}^q na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} = 1 + \frac{\frac{d_{q+1}}{q+1} \sum_{n=q+1}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \\ \left| \frac{g(z)-1}{g(z)+1} \right| &\leq \frac{\frac{d_{q+1}}{q+1} \sum_{n=q+1}^{\infty} n|a_n|}{2 - 2 \sum_{n=2}^q n|a_n| - \frac{d_{q+1}}{q+1} \sum_{n=q+1}^{\infty} n|a_n|}. \end{aligned} \tag{5.12}$$

$$\text{Now } \left| \frac{g(z)-1}{g(z)+1} \right| \leq 1 \text{ if } \sum_{n=2}^q n|a_n| + \frac{d_{q+1}}{q+1} \sum_{n=q+1}^{\infty} n|a_n| \leq 1. \tag{5.13}$$

Since the left hand side of(5.13)is bounded above by $\sum_{n=2}^q d_n|a_n|$ if

$$\sum_{n=2}^q (d_n - n)|a_n| + \sum_{n=q+1}^{\infty} d_n - \frac{d_{q+1}}{q+1} n|a_n| \geq 0. \tag{5.14}$$

and the proof is complete.

The result is sharp for the extremal function $u(z) = z + \frac{z^{q+1}}{d_{q+1}}$. \square

Theorem 5.3. *If $u(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$\Re \left[\frac{u'_q(z)}{u'(z)} \right] \geq \frac{d_{q+1}}{q + 1 + d_{q+1}}. \tag{5.15}$$

Proof. By setting

$$\begin{aligned} g(z) &= [q + 1 + d_{q+1}] \left[\frac{u'_q(z)}{u'(z)} - \frac{d_{q+1}}{q + 1 + d_{q+1}} \right] \\ &= 1 - \frac{\left(1 + \frac{d_{q+1}}{q+1} \right) \sum_{n=q+1}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^q na_n z^{n-1}} \end{aligned}$$

and making use of (5.14), we deduce that

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\left(1 + \frac{d_{q+1}}{q+1} \right) \sum_{n=q+1}^{\infty} n|a_n|}{2 - 2 \sum_{n=2}^q n|a_n| - \left(1 + \frac{d_{q+1}}{q+1} \right) \sum_{n=q+1}^{\infty} n|a_n|} \leq 1$$

which leads us immediately to the assertion of the Theorem 5.3. □

6. NEIGHBOUHOOD FOR THE CLASS $S^\xi(\tau, v)$

In this section, we determine the neighbourhoods for the class $S^\xi(\tau, v)$ which we define as follows:

Definition 6.1. A function $u \in A$ is said to be in the class $S^\xi(\tau, v)$ if there exist a function $g \in S(\tau, v)$ such that

$$\left| \frac{u(z)}{g(z)} - 1 \right| < 1 - v, \quad (z \in U, 0 \leq v < 1). \tag{6.1}$$

For any function $u(z) \in A, z \in U$ and $\delta \geq 0$, we define

$$N_{n,\delta}(u) = \left\{ g \in \Sigma : g(z) = z + \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\} \tag{6.2}$$

which is the (n, δ) -neighbourhood of $u(z)$.

The concept of neighbourhoods was first introduced by Goodman [8] and generalized by Ruscheweyh [16].

Theorem 6.2. If $g \in S(\tau, v)$ and

$$\xi = 1 - \frac{\delta(1 - v)}{2[(1 - v) - (1 + \tau)(2 + \rho - v)\Psi_2]} \tag{6.3}$$

then $N_{n,\delta}(g) \subset S^\xi(\tau, v)$.

Proof. Suppose $u \in N_{n,\delta}(g)$. We then find from (6.2) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \tag{6.4}$$

which yields the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2} \quad (n \in \mathbb{N}). \tag{6.5}$$

Next, since $g \in S(\tau, v)$, we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{(1 + \tau)(2 + \rho - v)\Psi_2}{1 - v}. \tag{6.6}$$

So that

$$\begin{aligned} \left| \frac{u(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \\ &= \frac{\delta(1 - v)}{2[(1 - v) - (1 + \tau)(2 + \rho - v)\Psi_2]} \\ &= 1 - \xi \end{aligned}$$

provided ξ is given by (6.3). Thus the proof of the is completed. □

CONCLUSION

The Hurwitz zeta function stands as a significant generalization of the Riemann zeta function, offering a richer framework for understanding complex analysis, number theory, and physics. Its properties, including analytic continuation, integral representations, and connections to special values, make it a versatile tool in mathematical research. With applications ranging from number theory and analytic number theory to theoretical physics and statistical mechanics, the Hurwitz zeta function serves as a bridge between various mathematical disciplines and provides insights into

the behavior of fundamental mathematical and physical phenomena. Its continued study and exploration promise to yield further discoveries and deepen our understanding of the intricate relationships between different areas of mathematics and physics.

REFERENCES

- [1] I. Aleksandar, *The Riemann Zeta-function: Theory and Applications*, John-Wiley and Sons, Inc., New York, 1985.
- [2] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. of Math.* 17 (1915), 12-22.
- [3] E.W. Barnes, The theory of the double gamma function, *Philos. Trans. Roy. Soc. A*, 196 (1901), 265-387.
- [4] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* 135 (1969), 429-446.
- [5] M. G. Bin-Saad, Hypergeometric series associated with the Hurwitz-Lerch zeta function, *Acta Math. Univ. Comenianae* (78), (2009), 269-286.
- [6] J. Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, *Appl. Math. Comput.*, 170 (2005), 399-409.
- [7] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, Toronto and London 1953.
- [8] A.W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, 8 (1957), 598-601.
- [9] I. B. Jung, Y. C. Kim, and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.* 176 (1993), 138-147.
- [10] R.J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, 135(1969), 429-449.
- [11] G. Murugusundaramoorthy, Subordination results for spirallike functions associated with Hurwitz-Lerch zeta function, *Integral Transforms and Special Functions* Vol. 23, Issue 2, (2012) 97-103.
- [12] N.M. Mustafa and M. Darus, Inclusion relations for subclasses of analytic functions defined by integral operators associated with the Hurwitz-Lerch Zeta function, *Tamsui Oxf. J. Math. Sci.* 28(4) (2012), 379-393.
- [13] H. Niranjana, A.N. Murthy, B. Venkateswarlu and P. Thirupathi Reddy, A new subclass of analytic functions defined by Lambda operator, *Palestine Journal of Mathematics*, 12(1), (2023), 2391-239.
- [14] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canadian Journal of Mathematics*. 39(5)(1987), 1057-1077.
- [15] I. Rabhaw and M. Darus On operator defined by double zeta functions, *Tamkang J. Math.* Vol.42,(2), (2011), 163-174.
- [16] S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, 81 (4) (1981), 521-527.
- [17] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* 51(1975), 109-116.
- [18] H. Silverman, Partial sums of starlike and convex functions, *J. Anal. Appl.*, 209 (1997), 221-227.
- [19] E.M. Silvia, Partial sums of convex functions of order α , *Houston J. Math.*, 11 (3) (1985), 397-404.
- [20] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
- [21] H. M. Srivastava and S. Owa, An application of the fractional derivative, *Mathematica Japonica*, 29(3)(1984), 383-389.
- [22] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transform. Spec. Funct.* 18 (2007), 207-216.
- [23] P. Thirupathi Reddy and B. Venkateswarlu, New subclass of analytic functions involving Hurwitz-Lerch Zeta functions, *International journal of Mathematics and Computation*, 31(1) (2020), 76-83.
- [24] B. Venkateswarlu, P. Thirupathi Reddy, G. Swapna and R.M. Shilpa, Certain classes of analytic functions defined by Hurwitz-Lerch Zeta function, *J. Appl. Anal.* 28(1); (2022), 73-81.
- [25] B. Venkateswarlu, P. Thirupathi Reddy, S. Sridevi and Sujatha, On a certain subclass of analytic functions defined by Geanbauer polynomials, *Palestine Journal of Mathematics*, 12(1), (2023), 187-196.