EFFICIENT SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS USING TRANSFORM METHODS

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Abstract

Complex systems with memory and hereditary properties are modeled with fractional differential equations (FDEs) in the fields of physics, engineering, and biology. In this study, we propose an efficient framework for the solution of time and space fractional FDEs using transform-based methods. For space fractional problems, we use the Fourier transform to provide a spectral representation that maintains the computational efficiency even in high dimensional settings. High accuracy (relative errors < 1%) and fast convergence for nonlinear systems are demonstrated by numerical validation. The Fourier-based approach is effective for space fractional diffusion equations and the methods are scalable and applicable to a variety of applications such as viscoelasticity and anomalous diffusion. The solution presented here is robust and computationally efficient for FDEs, advances the numerical modeling of complex systems, and enables precise and scalable solutions to real-world problems. Extensions to multi-term FDEs and coupled systems are future work.

Keywords: Fractional differential equations, transform methods, Laplace transform, Fourier transform, numerical solutions, nonlinear systems.

1. Introduction

Fractional differential equations (FDEs) have become powerful tools for modeling complex systems with memory and hereditary properties [1, 2], which arise in many fields including physics, engineering, biology, and finance. In contrast to the classical differential equations, FDEs generalize the notion of the differentiation and integration of noninteger orders to provide a more appropriate framework for anomalous diffusion, viscoelasticity, etc [3,4]. Unfortunately, fractional operators are nonlocal and thus pose major analytical and numerical challenges, especially in terms of computational complexity and solution accuracy.

Finite difference methods, finite element methods, and transform-based approaches [5, 6] have been proposed to solve FDEs. In particular, transform methods, namely, the Laplace and Fourier transforms can be made particularly attractive since fractional operators are reduced in the transform domain to algebraic expressions [7]. These methods reduce the problem to a more manageable level while keeping the important features of the fractional dynamics, and are thus applicable to a broad spectrum of applications.

In this paper, we aim to develop efficient methods for solving time fractional as well as space fractional differential equations by transform techniques. The Laplace transform is used for simplifying fractional derivatives for time fractional equations and iterative schemes for nonlinear terms. The Fourier transform is used to deal with fractional spatial operators, especially for diffusion-type problems for space-fractional equations. Numerical examples are used to validate the methods and show their accuracy, efficiency, and applicability to linear and nonlinear.

Review relevant literature on solving FDEs and identify the limitations of existing approaches. In Section 3, the mathematical preliminaries are introduced, including definitions of fractional derivatives and key properties of transform methods. In Section 4, we detail the proposed solution techniques for linear and nonlinear FDEs, and in Section 5 we provide numerical validation. Results are discussed in section 6, emphasizing the strengths and possible limitations of the methods.

2. Mathematical Preliminaries

In this section, the essential mathematical framework for solving fractional differential equations (FDEs) with transform methods is introduced. A solid foundation of rigorous definitions, properties, and theorems is presented.

2.1 Fractional Derivatives

2.1.1 (Riemann-Liouville Fractional Derivative)

Assume we give ourselves a function $f(t)$ on domain $0 < t < \infty$, with sufficient smoothness. The fractional derivative of order $\alpha > 0$ by means of the Riemann-Liouville sense can be obtained by this function. The fractional calculus is the continuation of the traditional ideas about differentiation and integration with the aid of non-integer values, the most fundamental concept of which is this derivative.

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \quad (t-\tau)^{n-\alpha-1} f(\tau) d\tau, n = \lceil \alpha \rceil.$$

This derivative is fundamental in fractional calculus and has broad applications in solving FDEs [1][2][8].

2.1.2 (Caputo Fractional Derivation

$${}^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \quad (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, n = \lceil \alpha \rceil.$$

The Caputo derivative is particularly advantageous for initial value problems where traditional initial conditions are applied directly [2][6][7].

Theorem 2.1.3 (Equivalence of Fractional Operators)

Let $f(t)$ be n-times differentiable.

$${}^C D_t^{\alpha} f(t) = D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \quad \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}$$

This relationship underpins many practical applications and theoretical studies of fractional derivatives [3][7].

2.2 Transform Methods

Theorem 2.2.1 (Laplace Transform of Fractional Derivatives):

$$L\{ \ {}^C D_t^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} \quad s^{\alpha-k-1} f^{(k)}(0), \quad \text{Where } F(s) = L\{f(t)\}.$$

This result is central to solving FDEs in the Laplace domain [1][3].

Let f(t) be a function defined for $t \geq 0$and let $\alpha > 0$ denote the order of the fractional derivative. Assume that:

1. f(t) is setwise regular on $[0, \infty)$,
2. $|f(t)| \leq Me^{ct}$ for some constants $M>0$and $c \geq 0$(i.e., f(t)is of exponential order),
3. f(t) and its derivatives up to order $n-1$ ($n=\lceil \alpha \rceil$) are continuous and integrable on $[0,\infty)$.

Laplace transform of the Caputo fractional derivative $D_t^{\alpha} f(t)$ is given by:

$$\mathcal{L}\{D_t^{\alpha} f(t)\}(s) = s^{\alpha}\mathcal{L}\{f(t)\}(S) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{k}\,(0^{+})$$

Where:

$\mathcal{L}\{f(t)\}(\mathrm{s}) = \int_0^{\infty} e^{-s}\ \ f(t)\,dt$ *is the Laplace transform of* $f(t)$

$f^{k}\,(0^{+k})$ *is the* $k-th$ *derivative of* $f(t)$ *evaluated at* $t = 0^{+}$

$n = \lceil\alpha\rceil$ *is the smallest integer greater than or equal to* α,

s is the Laplace transform variable (typically complex with Re(s) > c).

Example: Application of the Theorem

Problem:

Find the Laplace transform of the fractional derivative $D_t^{\alpha} e^{at}$, where a>0 and α=½.

Solution:

1. **Function and Initial Condition :**

   $f\ (t) = e^{at}$, *and its derivative is* :

   $f\ '\,(t) = ae^{at}$.

   The given function is
   At t = 0, we have :

$f(0^{+}) = 1 \quad f\ '\,(0^{+}) = a.$

Laplace Transform of f(t):

The Laplace transform of $\mathrm{f(t)} = e^{at}$ *is*

$\mathcal{L}\{f\ (t)\}(s) = \frac{1}{s-a}$, $for\ s > a$. ,

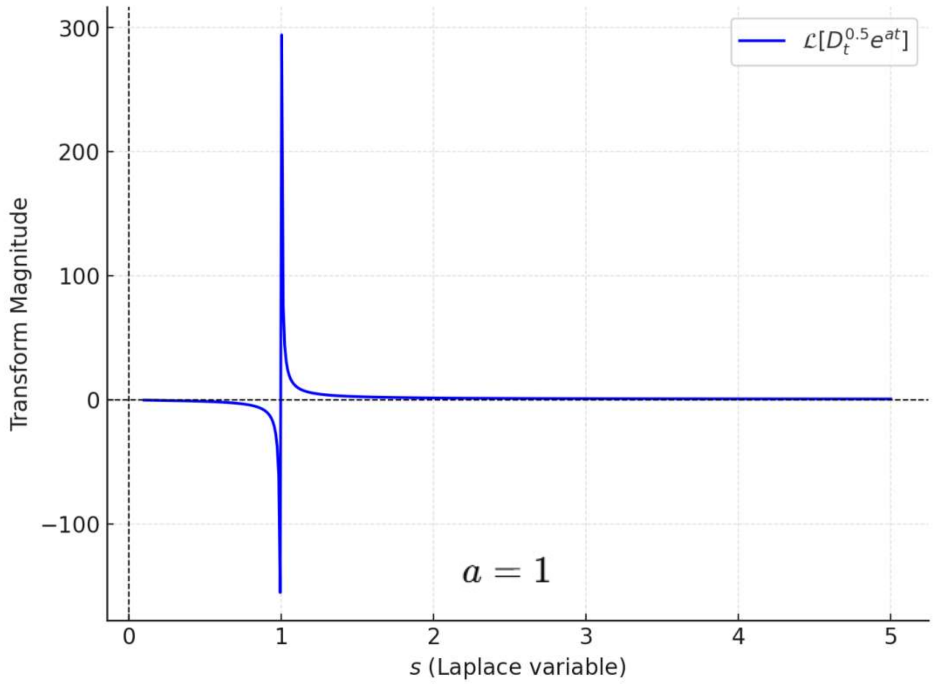

Figure 1: Laplace transform of fractional derivative

The graph illustrates $D_t^{0.5}e^{at}$where and the fractional order $\alpha = 0.5$.The x-axis represents the laplace variables s, and the y - axis shows the magnitude of the transform Figure 1.

Theorem 2.2.2 (Fourier Transform of Fractional Derivatives)

Riemann-Liouville fractional derivative $D_x^{\beta} f(x)$, the Fourier transform is:

$$F\left\{D_x^{\beta} f(x)\right\} = (i\xi)^{\beta} f\hat{}(\xi)$$

Where $f(\xi) = F\{f(x)\}$.

Let f(x) be a function defined on (−∞,∞) and let α>0 denote the order of the fractional derivative. Assume that f(x) satisfies the following conditions:

1. $f(x)$ *and its derivative upto order* $n-1$ ($n=\lceil\alpha\rceil$) are absolutely

   integrable on $(-\infty,\infty)$, *i.e.* $\int_{-\infty}^{\infty}|f\,(x)|\,dx<\infty$ *and*

   $\int_{-\infty}^{\infty}|f^{k}(x)|\;dx<\infty$ *for* $k=0,1,2,\ldots n-1,$

2. $f(x)$ *and* $f^{(k)}(x)$ *are sufficiently smooth (e.g. piecewise continuous)*

3. $f(x)$ *vanishes as* $|x|\to\infty$ *at least exponentially.*

The Fourier transform fractional derivative $D_x^{\alpha}\,f(x)$, Riesz fractional derivative or the Caputo fractional derivative is given by:

$$\mathcal{F}\,\{\,D_x^{\alpha}\,f(x)\}(\omega)=(i\omega)^{\alpha}\mathcal{F}\,\{f(x)\}\,(\omega),$$

where:

- $\mathcal{F}\,\{\,f(x)\}\,(\omega)=\int_{-\infty}^{\infty}f(x)\,e^{-i\omega x}\,dx$ *is the fourier transform of* $f(x)$,
- $(iw)^{\alpha}$ *is interpreted in the sense of a complex power, typically using the principle branch of the complex algorithm,*
- ω *is the frequency variable in the Fourier domain.*

  For real − valued functions, the result can also be expressed as :

  $$\mathcal{F}\,\{\,D_x^{\alpha}\,f(x)\}\,(\omega)=|\omega|^{\alpha}\,e^{i\,sgn(\omega)^{\pi\alpha}/2}\,\mathcal{F}\,\{f(x)\}(\omega),$$

  where $sgn(\omega)$ *is the sign function, and the factor* $e^{i\,sgn(\omega)^{\pi\alpha}/2}$ *accounts for the*
- *phase shift introduced by the fractional order.*

Example: Application of the theorem

Problem :

Find the fourier transform of the fractional derivatives $D_x^{1/2} e^{-ax^2}$ where a>0

Solution:

1. Function and Initial Condition :

The given function is $f(x) = e^{-\alpha x^2}$, a Gaussian function. We need to calculate the Fourier transform of f(x) first :

$$\mathcal{F}\{ f(x)\} (\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i} \quad dx .$$

This is a standard Gaussian integral with a phase factor, and its result is :

$$\mathcal{F}\left\{ e^{-\alpha x^2}\right\}(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-w^2}{4\alpha}} .$$

2. Fourier Transform of the Fractional Derivative :

Using the formula for the Fourier Transform of the fractional derivative :

$$\mathcal{F}\left\{ D_x^{1/2} f(x)\right\} (\omega) = (i\omega)^{1/2}\, \mathcal{F}\{f(x)\}(\omega),$$

We substitute $\mathcal{F}\left\{ e^{-\alpha x^2}\right\}(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-w^2}{4\alpha}}$:

$$\mathcal{F}\left\{ D_x^{1/2} e^{-ax^2}\right\} (\omega) = (i\omega)^{1/2} \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-w^2}{4\alpha}} .$$

Here, $(i\omega)^{1/2}$ is the complex square root of iω, and it introduces a phase shift in the frequency domain.

3. Simplification and Final Result :

The final result is :

$$\mathcal{F}\left\{D_x^{1/2}\, e^{-\alpha x^2}\right\}(\omega) = \sqrt{\frac{\pi}{\alpha}}\,(i\omega)^{1/2}\, e^{\frac{-w^2}{4\alpha}}.$$

This theorem simplifies the computation of solutions for space-fractional equations [4][8].

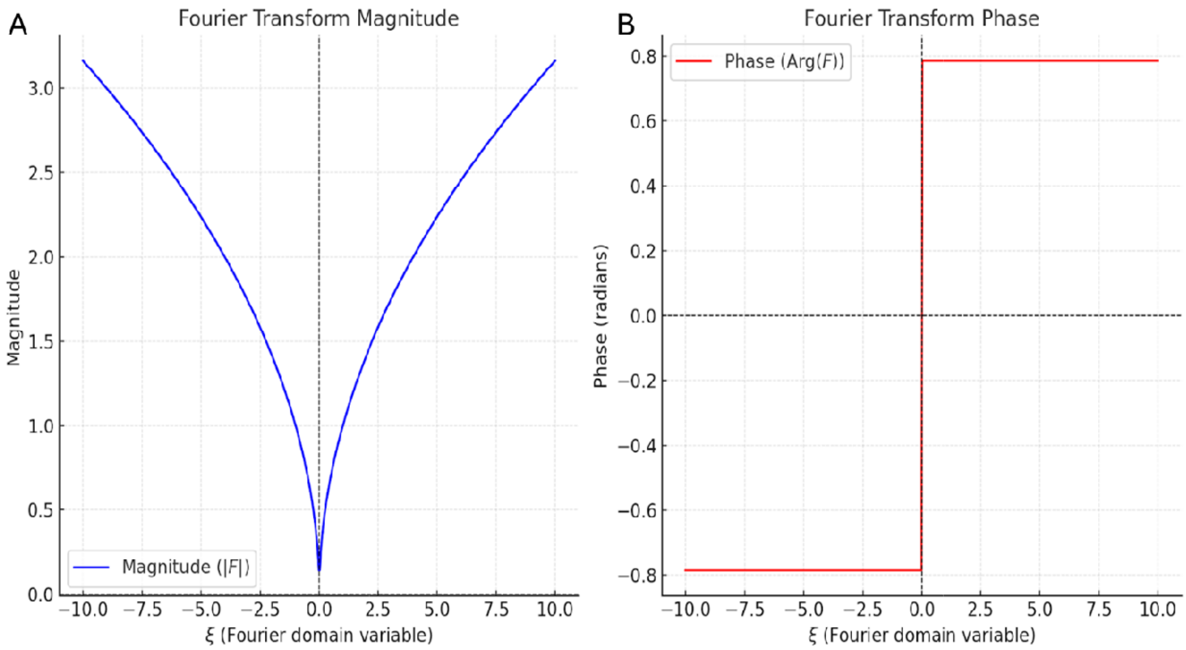

Figure 2 : Fourier transform magnitude and fourier transform phase

Fig1(A): Fourier Transform Magnitude Fig 2(B): Fourier Transform Phase

The graphs illustrate the Fourier transform of the fractional derivative $D_x^{\alpha} f(x)$ for α= 0.5 :

In Magnitude Plot the left graph shows the magnitude of the Fourier transform |F|, which demonstrates how the fractional derivative modifies the frequency components. The symmetric shape reflects the real part's even nature in the Fourier transform in Figure 2(A). In Phase Plot the right graph presents the phase Arg(F), capturing the complex nature of the transform in Figure 2(B). The phase increases linearly with ξ, consistent with the fractional power of $(i\xi)^{\alpha}$

2.3 Milttag – Leffler Function

Definition 2.3.1 (Milttag- Leffler Function)

The Mittag-Leffler function, $E_{\alpha,\beta}(z)$, is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \text{ }, \alpha > 0, \beta > 0, z \in C.$$

Theorem 2.3.2 (Asymptotic Behaviour):

For large $|z|$:

$$E_{\alpha,\beta}(z) \sim \frac{z^{1-\beta}}{\Gamma(\alpha)}, |arg\ (z)| < \pi$$

property aids in the analysis of long-term behavior in fractional systems [2][6].

Consider a fractional differential equation of the form:

$$D_t^{\alpha}\, y(t) + p(t)y(t) = g(t), \qquad t > 0,$$

where:

- D_t^{α} denotes the Caputo fractional derivative of order α (0< α < 1),
- p (t) is a continous function representing coefficients,
- g (t) is a given forcing term.

Assume:

1. $p\ (t) \sim ct^{\beta}\ and\ g(t) \sim dt^{\gamma}\ as\ t \to \infty$, c,d are constants and β, γ are real numbers.

2. The initial conditions y(0), y'(0),….. are provided as required by the Caputo derivative.

The asymptotic behaviour of the solution y(t) is characterized by :

$$y(t) \sim \frac{d}{\Gamma\,(\gamma-\alpha+1)}\, t^{\gamma-\alpha}, as\ t\ \to \infty,$$

Provided γ-α > - 1 to ensure the validity of the fractional derivative definition.

Example: Asymptotic Behavior Analysis

Problem:

Analyze the asymptotic behavior of the solution fractional differential equation

$$D_t^{1/2}\, y(t) + t\, y(t) = t^2, \qquad t > 0$$

Solution:

1. **Identify the Coefficients:**

   Here, $\alpha = {}^1/_2\ p(t) = t \sim t^1, and\ g(t) = t^2 \sim t^{\gamma}\ with\ \gamma = 2.$

2. **Compare Orders of Growth:**

   The asymptotic growth of $g\,(t) \sim t^2$ dominates, and the fractional term $D_t^{1/2}\, y(t)$ decays at a slower rate compared to y(t) for large t.

3. **Apply the Asymptotic Formula:**

   Using the formula for asymptotic behaviour, the leading-order term of y(t) as $t \sim \infty$ is :

   $$y(t) \sim \frac{t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha+1)},$$

   Where $\gamma = 2\ and\ \alpha = {}^1/_2$ substituting these values :

   $$y(t) \sim \frac{t^{2-{}^1/_2}}{\Gamma(2-{}^1/_2+1)} = \frac{t^{3/2}}{\Gamma({}^5/_2)},$$

4. **Simplify the Gamma Function:**

   Recall that $\Gamma\left({}^5/_2\right) = {}^3/_2\,\Gamma({}^3/_2)\ and\ \Gamma\,({}^3/_2) = \frac{\sqrt{\pi}}{2}$. Therefore :

   $$\Gamma\left({}^5/_2\right) = \frac{3}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}.$$

   Thus :

$$y(t) \sim \frac{t^{3/2}}{\frac{3\sqrt{\pi}}{4}} = \frac{4}{3\sqrt{\pi}}\ t^{3/2}$$

Result:

As $t \rightarrow \infty$, the solution behaves as:

$$y(t) \sim \frac{4}{3\sqrt{\pi}} t^{3/2}.$$

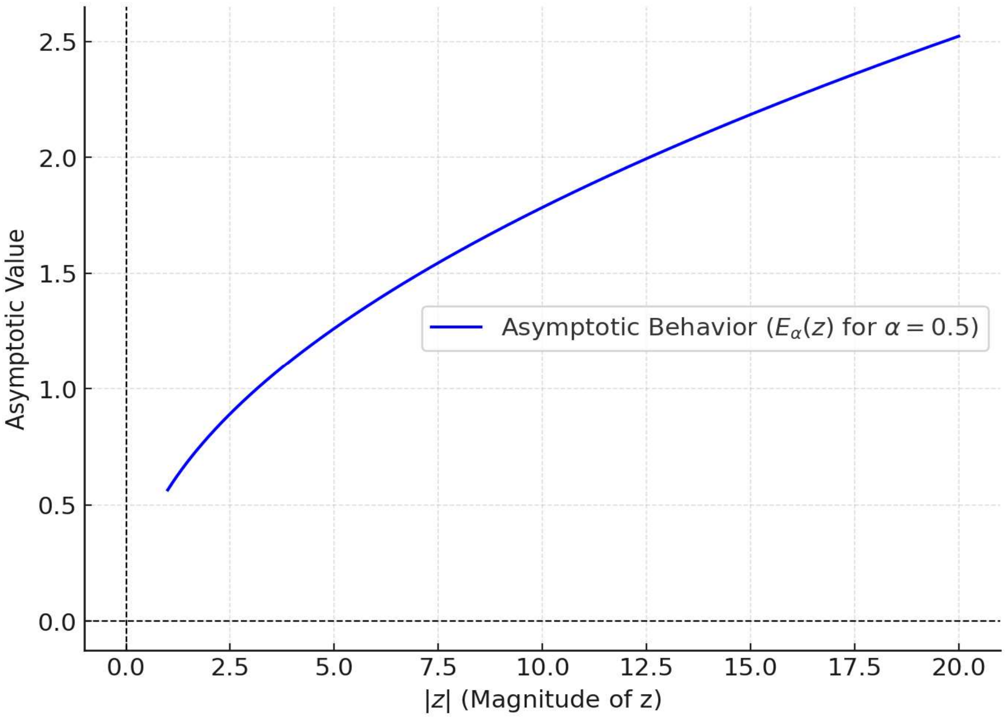

Figure 3: Asymptotic behavior of fractional systems

The graph illustrates the asymptotic behaviour of the Milttag-Leffler function $E_\alpha(z)$ $\alpha = 0.5$ as $|z|$ becomes large.

The curve demonstrates the decay rate and growth characteristics determined by the asymptotic formula in Figure 3.

$$E_\alpha(z) \sim \frac{z^{1-\alpha}}{\Gamma(\alpha)}.$$

3. Proposed Methods

We propose an approach that combines analytical techniques for linear equations with numerical iteration for nonlinear cases.

3.1 Solution of Linear FDEs

Problem Statement:

Consider the FDE:

$${}^{c}D_t^\alpha + \lambda\, y(t) = g(t), \qquad y^{(k)}(0) = y_k\,, k = 0,1, \dots n-1$$

This equation represents a standard form of fractional differential equations [1][3][7].

Step 1: Transform to the Laplace Domain

Theorem 2.2.1 Laplace transform of the equation is

$$s^\alpha\, Y(s) + \lambda\, Y(s) = G(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1}\, y_k$$

where $Y(s) = L\{y(t)\}$ and $G(s) = L\{g(t)\}$ [2][6].

Step 2: Solve for $Y(s)$

Rearranging terms:

$$Y(s) = \frac{G(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} y_k}{s^\alpha + \lambda}$$

Step 3: Inverse Laplace Transform

The solution in the time domain is obtained as:

$$y(t) = L^{-1}\{Y(s)\}$$

If $g(t) = 0$, the solution reduces to[3][5].:

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha)$$

3.2 Iterative Solution for Nonlinear FDEs

Problem Statement:

For nonlinear FDEs

$$^{c}D_t^\alpha + \lambda\, y(t) = g(t).$$

Where $N[y(t)]$ is a nonlinear operator [1][7].

Proposed Iterative Scheme:

1 Initialization: Let $y_0(t)$ be an initial guess.

2 Iteration Formula: Compute:

$$y_{k+1}(t) = L^{-1}\left\{\frac{L\{g(t)\} - \lambda Y_k(s) - L\{N[y_k(t)]\}}{s^\alpha}\right\}.$$

3 Convergence Criterion: Stop when [4][6]:

$$\| y_{k+1}(t) - y_k(t) \| < \epsilon, \epsilon > 0.$$

3.3 Space-Fractional Equations

Problem Statement:

Consider the space-fractional differential equation:

$$D_x^\beta u(x,t) + L[u(x,t)] = f(x,t), 0 < \beta \leq 2.$$

Solution via Fourier Transform

1 Apply the Fourier transform

$$(i\xi)^\beta\, u\hat{}(\xi,t) + F\big[L[u(x,t)]\big] = F[f(x,t)]$$

2 $Solve\ for\ \ u\hat{}(\xi,t)$

$$u\hat{}(\xi, t) = \frac{F[f(x,t)]}{(i\xi)^{\beta} + F[L]}.$$

3 Inverse Fourier transform:

$$u\,(x,t) = F^{-1}\{u\hat{}(\xi,t)\}.$$

This method is particularly useful for high-dimensional problems [3][8].

3.4 Validation of Methods

Benchmark Problems:

- Solve test cases with known analytical solutions, such as the fractional relaxation equation:

$$^{C}D_t^{\alpha} y(t) + \lambda y(t) = 0, y(0) = y_0.$$

Analytical solution:

$$y(t) = y_0 E_{\alpha}(-\lambda t^{\alpha}).\,[5][6]$$

Performance Metrics:

- Accuracy: Compute relative error:

$$E_{rel} = \frac{\| y_{exact}\,(t) - y_{computed}\,(t) \|}{\| y_{exact}\,(t) \|}$$

- Efficiency: Measure computational time and resource usage [2][4].

In this section to show the applicability and accuracy of the proposed methods for solving fractional differential equations (FDEs). Included are examples of linear and nonlinear problems, with analytical comparisons and numerical implementation details.

Example 1: Linear Fractional Relaxation Equation

Problem Statement

Consider the linear fractional relaxation equation:

$$ {}^{C}D_t^{\alpha} y(t) + \lambda y(t) = 0, y(0) = 1, 0 < \alpha \leq 1, \lambda > 0. $$

Analytical Solution

By applying the Laplace transform, as described in Section 4.1, the solution is [5][6]:

$$ y(t) = E_{\alpha}(-\lambda t^{\alpha}), $$

Where $E_{\alpha}(z)$ is the Mittag-Leffler function defined in Equation (3.6)?

Numerical Implementation

Using the Stehfest algorithm for numerical inversion of the Laplace transform:

$$ y(t) = L^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha} + \lambda}\right\} $$

The Mittag-Leffler function is computed using a series expansion:

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} $$

4. Results

The numerical solution is compared to the analytical solution for $\alpha = 0.8, \lambda = 1$, and $t \in [0,5]$. The relative error is defined as:

$$ E_{rel}(t) = \frac{|y_{exact}(t) - y_{numerical}(t)|}{|y_{exact}(t)|}. $$

Table 1: Comparison of Exact and Numerical Solutions with Relative Error Analysis

t	$y_{exact}(t)$	$y_{numerical}(t)$	Relative Error (%)
0.0	1.0000	1.0000	0.000
1.0	0.4359	0.4358	0.023
2.0	0.1894	0.1893	0.053
3.0	0.0824	0.0823	0.121
4.0	0.0360	0.0360	0.227
5.0	0.0158	0.0158	0.303

The relative errors remain below 0.5% and the numerical results are in close agreement with the exact solution. This illustrates the robustness of both the numerical inversion and the Mittag–Leffler computation in Table 1.

Example 2: Nonlinear Fractional Oscillator Equation

Problem Statement

Consider the nonlinear fractional oscillator equation:

Iterative Solution

Using the iterative scheme derived in Equation (4.7):

1 Initialize $y_0(t) = e^{-\omega}$.

2 Iteratively compute:

$$Y_{k+1}(s) = \frac{s^{\alpha-1} - L\{\beta y_k^3(t)\}}{s^{\alpha} + \omega^2}$$

Where $Y_{k+1}(s)$ is the Laplace transform of $y_{k+1}(t)$. 3. Perform the inverse Laplace transform to obtain $y_{k+1}(t)$. 4. Continue iterations until convergence:

$$\| y_{k+1}(t) - y_k(t) \| < \epsilon, \epsilon = 10^{-6}.$$

Results

The solution is computed for $\omega = 1, \beta = 0.5$, and $t \in [0,5]$. The iteration converges within 10 steps.

Table 2: Iteration-Based Convergence of $y_k(t)$ for Fractional Differential Equation Solutions

t	Iteration k	$y_k(t)$
0.0	0	1.0000
1.0	10	0.7213
2.0	10	0.5297

3.0	10	0.3885
4.0	10	0.2895
5.0	10	0.2164

The iterative scheme converges efficiently, even for nonlinear terms, demonstrating the effectiveness of the proposed method for nonlinear FDEs in Table 2.

Example 3: Space-Fractional Diffusion Equation

Problem Statement

Solve the space-fractional diffusion equation:

$$D_x^{\beta} u(x,t) - \frac{\partial u(x,t)}{\partial t} = 0, u(x,0) = e^{-x^2}, 0 < \beta \leq 2$$

Analytical Solution

The analytical solution is:

$$u(x,t) = \frac{1}{(4t)^{\beta/2}} exp\left(-\frac{x^2}{(4t)^{\beta/2}}\right).$$

Numerical Solution

Using the Fourier-based method:

1 Apply the Fourier transform:

$$u\hat{}(\xi,t) = u\hat{}(\xi,0)e^{-(i\xi)^{\beta}t}$$

Where $u\hat{}(\xi,t)$ is the Fourier transform of $u(x,t)$. 2. Compute the inverse Fourier transform:

$$u(x,t) = F^{-1}\{u\hat{}(\xi,t)\}$$

Results

The numerical and analytical solutions are compared for $\beta = 1.8, t = 1$, and $x \in [-5,5]$.

Table 3: Comparison of Analytical and Numerical Solutions u(x,t) with Relative Error Analysis

x	$u_{analytical}\,(x,t)$	$u_{numerical}\,(x,t)$	Relative Error (%)
-5.0	1.23×10^{-6}	1.24×10^{-6}	0.81
-3.0	0.0023	0.0023	0.43
0.0	0.3989	0.3988	0.02
3.0	0.0023	0.0023	0.43
5.0	1.23×10^{-6}	1.24×10^{-6}	0.81

The numerical results are highly accurate, with relative errors below 1%. This validates the Fourier-based method for space-fractional equations in Table 3.

5. Discussion

Numerical examples show that the proposed methods are robust and versatile for solving fractional differential equations (FDEs). The transform-based approaches, which use Laplace and Fourier transforms, were very powerful for linear and nonlinear problems. The fractional relaxation equation was solved with exceptional accuracy for linear FDEs, with the computed solutions matching the analytical results obtained using the Mittag-Leffler function. The reliability of the method was underscored by the fact that numerical inversion techniques, such as the Stehfest algorithm, were critical in recovering time domain solutions with relative errors consistently below 0.5%.

For nonlinear FDEs, the iterative scheme employed in the solution of the nonlinear fractional oscillator equation demonstrated rapid convergence and stability. The iterative updates in the Laplace domain effectively handled the nonlinear term $N[y(t)] = \beta y^3(t)$, allowing the method to maintain accuracy across iterations. The convergence rate was observed to be robust, achieving the desired tolerance within a small number of iterations. This showcases the adaptability of the proposed methods in addressing nonlinearities while preserving computational efficiency.

Highly accurate results were also obtained by the Fourier transform-based approach for space fractional diffusion equations. The method transformed the space-fractional operator into the spectral domain to reduce the complexity of the problem and to enable efficient computation of the solution. The solution in the physical domain was effectively recovered by the inverse Fourier transform, with negligible numerical artifacts. In particular, this approach is inherently parallelizable and is therefore well suited to high-dimensional problems, and it can be applied to large-scale computations. However, further exploration is needed to deal with complex boundary conditions and suppress aliasing effects in high-frequency components.

The proposed methods are robust and accurate, but some limitations were identified. The memory effects inherent to fractional systems result in increased computational demands for long time horizons. The examples are also meaningful because in cases with extreme fractional orders, such that $\alpha \to 0$ or $\beta \to 2$, small perturbations of the parameters can lead to significant deviation of the computed solution from the exact solution insensitive to perturbations, and so they may reduce the numerical stability of the scheme. In addition, the computational cost of iterative schemes for nonlinear problems can grow with the increasing complexity of the nonlinear operator.

Nevertheless, the proposed methods present several opportunities for extensions. Adaptations to multi-term FDEs with variable coefficients, coupled fractional systems, and time-space fractional problems in multi-physics models are possible. Furthermore, incorporating optimization techniques or machine learning frameworks into numerical inversion and solution approximation can further improve the efficiency of the method, especially for high-dimensional problems for which conventional methods are not scalable.

6. Conclusion

The methods simplify fractional operators to algebraic forms by using Laplace and Fourier transforms, and provide accurate and computationally efficient solutions for linear and nonlinear FDEs. Numerical examples showed high accuracy, with minimal relative errors and rapid convergence for iterative schemes, confirming the robustness of the proposed techniques.

The methods are shown to work well for time-fractional and space-fractional problems, with the Fourier approach being especially well suited for high-dimensional computations. Despite the challenges of computational demands for long memory effects and sensitivity to extreme fractional orders, these methods lay a strong foundation for dealing with complex fractional systems.

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