GENERALIZED DIFFERENTIAL OPERATOR DEFINED BY q- BOREL DISTRIBUTION

Katterapalle Sridevi
Department of Mathematics,
Dr.B.R.Ambedkar Open University, Hyderabad - 500 033, T.S, India.

N.Sri Lakshmi Sudha Rani
Department of Mathematics,
Dr.B.R.Ambedkar Open University, Hyderabad - 500 033, T.S, India.

Abstract: The **q-calculus** is a generalization of traditional calculus that introduces a parameter q typically where $0 < q < 1$ and it generalizes concepts like differentiation and integration using a parameter-dependent approach. It plays a significant role in various areas such as combinatorics, special functions, quantum groups, and more. The target of this paper is to define the operator of q- derivative based upon the Borel distribution and by using this operator, we obtain the coefficient bounds, inclusion relations, extreme points and some more properties of defined class.

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1. Inroduction.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk U = {z : |z| < 1} and normalized by $f(0)=0, f'(0)=1$. Let S be the subclass of A consisting of univalent functions $f(z)$ of the form (1.1). Further denote by T the subclass of A consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n, (a_n \geq 0) \tag{1.2}$$

introduced and studied by Silverman [7].

For $g(z)=z+\sum_{n=2}^{\infty}b_n z^n$, the Hadamard product (or convolutions) of *f* and *g* is defined by

$$(f*g)\,z=z+a_n b_n z^n,\ z\in U \tag{1.3}$$

The elementary distribution such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [1,2,5,6].

A discrete random variable x is said to have a Borel distribution if it takes the values $1,2,3,\cdots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \cdots$, respectively, where λ is called the parameter. Very recently, Wanas and Khuttar [10] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=\varrho)=\frac{(\varrho\lambda)^{\varrho-1}e^{-\lambda\varrho}}{\varrho!}, \varrho=1,2,3,\cdots$$

Wanas and Khuttar [10] introduced a series $M(\lambda;z)$ whose coefficients are probabilities of the Borel Distribution (BD)

$$\begin{aligned} M(\lambda;z) &= z+\sum_{k=2}^{\infty}\frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!}z^k, \quad (0<\lambda\leq 1) \\ &= z+\sum_{k=2}^{\infty}\sigma_k(\lambda)z^k, (0<\lambda\leq 1), \end{aligned} \tag{1.4}$$

where

$$\sigma_k(\lambda) = \frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!}.$$

We define a linear operator $\mathfrak{B}(\lambda; z)f: A \to A$ as follows:

$$\begin{aligned} \mathfrak{B}(\lambda; z)f(z) &= M(\lambda; z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!} a_k z^k, \ \ (0 < \lambda \leq 1). \end{aligned}$$

Srivastava [9] made use of various operators of q- calculus and fractional q- calculus and recalling the definition and notations. The q- shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup 0$ as follows:

$$(\lambda; q)_k = \begin{cases} 1, & \text{for } k = 0, \\ (1-\lambda)(1-\lambda q) \cdot (1-\lambda q^{k-1}), & \text{for } k \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda}; q\right)_k = \frac{(1-q)^k \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)} \quad (k \in \mathbb{N}_0),$$

where (see [9])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} \quad (|q| < 1).$$

Also, we note that

$$(\lambda; q)_{\infty} = \prod_{k=0}^{\infty} (1 - \lambda q^k) \quad (|q| < 1)$$

and, the q- gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z),$$

where$[k]_q$ denotes the basic q- number defined as follows:

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & \text{for } k \in \mathbb{C}, \\ 1 + \sum_{j=1}^{k-1} q^j, & \text{for } k \in \mathbb{N}. \end{cases} \tag{1.5}$$

Using the definition formula (1.5) we have the next two products:

(i) For any non-negative integer k, the q-shifted factorial is given by

$$[k]_q! := \begin{cases} 1, & \text{for } k = 0, \\ \prod_{n=1}^{k} [n]_q, & \text{for } k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the q- generalized Poccammer symbol is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{for } k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & \text{for } k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \quad \text{as} q \to 1^-$$

Also, we observe that

$$\lim_{q\to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k.$$

For $0 < q < 1$, the q- derivative operator [9] (see also [10]) for $\mathfrak{B}(\lambda; z)f$ is defined by

$$\begin{aligned} D_q\big(\mathfrak{B}(\lambda; z)f(z)\big) &:= \frac{\mathfrak{B}(\lambda; z)f(z) - \mathfrak{B}(\lambda; z)f(qz)}{z(1-q)} \\ &= 1 + \sum_{k=2}^{\infty} [k]_q \frac{[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)!} a_k z^{k-1}, \quad (0 < \lambda \leq 1,\ z \in E), \end{aligned}$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \quad [0, q] := 0.$$

For $\vartheta > -1$ and $0 < q < 1$, we defined the linear operator $\mathfrak{B}_{\lambda}^{\vartheta,q} f: A \to A$ by

$$\mathfrak{B}_{\lambda}^{\vartheta,q} f(z) * N_{q,\vartheta+1}(z) = zD_q\big(\mathfrak{B}(\lambda; z) f(z)\big), \quad z \in E,$$

where the function $N_{q,\vartheta+1}$is given by

$$N_{q,\vartheta+1(z)} := z + \sum_{k=2}^{\infty} \frac{[\vartheta+1]_{q,k-1}}{[k-1]_q!} z^k, \quad z \in E.$$

A simple computation shows that

$$\begin{aligned} \mathfrak{B}_{\lambda}^{\vartheta,q} f(z): \quad &= z + \sum_{k=2}^{\infty} \frac{[k]_q!\,[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{[\vartheta+1]_{q,k-1}(k-1)!} a_k z^k \\ &= z + \sum_{k=2}^{\infty} \mathcal{D}(k)\ a_k z^k, \end{aligned} \tag{1.6}$$

where

$$\mathcal{D}(k) = \frac{[k]_q!\,[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{[\vartheta+1]_{q,k-1}(k-1)!} \tag{1.7}$$

and $0 < \lambda \le 1, \vartheta > -1, 0 < q < 1, z \in E$.

Now using above differential operator, we define the following subclass of T.

Definition 1.1. A function $f \in \mathcal{A}$ given by (1.2) is in the class $\mathcal{T}_q(\lambda, \beta, \vartheta, A, B)$, $(0 < q < 1), \lambda \in \mathbb{N}_0, \beta \ge 0, \vartheta > -1,\ and\ -1 \le A < B \le 1, 0 < B \le 1)$ if it satisfies the following subordination condition:

$$(1-\beta)\frac{\mathfrak{B}_{\lambda}^{\vartheta,q} f(z)}{z} + \beta\left(\mathfrak{B}_{\lambda}^{\vartheta,q} f(z)\right)' \prec \frac{1+Az}{1+Bz} (z \in \mathrm{E}). \tag{1.8}$$

Silverman[8]introduced and studied the univalent function with varying arguments of coefficients as follows:

Definition 1.2. [8] A function $f(z)$ of the form (1.1) is in the class $\mathcal{V}(\theta_k)$ if $f(z) \in \mathcal{S}$ (the class of analytic and univalent function in) and $\arg(a_k) = \theta_k$ for all $k(k \geq 2)$. Further, if there exists a real number η such that

$$\theta_k + (k-1)\eta \equiv \pi(\mathrm{mod} 2\pi), \qquad (1.9)$$

then $f(z)$ is said to be in the class $\mathcal{V}(\theta_k, \eta)$. The union of $\mathcal{V}(\theta_k, \eta)$ taken over all possible sequence $\{\theta_k\}$ and all possible real numbers η is denoted by $\mathcal{V}$.

Let $\mathcal{V}_q(\lambda, \beta, \vartheta, A, B)$ denote the subclass of $\mathcal{V}$ consisting of functions $f(z) \in \mathcal{T}_q(\lambda, \beta, \vartheta, A, B)$.

In this paper, the authors obtain coefficient estimates, distortion theorem and extreme point for the function $f \in \mathcal{A}$ belongs to the class $\mathcal{V}_q(\lambda, \beta, \vartheta, A, B)$.

2.1 Coefficient Estimates

Unless otherwise stated, we assume throughout the sequel that $-1 \leq A < B \leq 1, 0 < B \leq 1, \lambda, \beta \in \mathbb{N}_0, 0 < q < 1; z \in$ E.

The sufficient condition for a function $f(z)$ of the form (1.2) to be in the class $\mathcal{T}_q(\lambda, \beta, \vartheta, A, B)$ is given by the following theorem.

Theorem 2.1. Let the function $f(z)$ be of the form (1.2). If

$$\sum_{k=2}^{\infty} [1 + \beta(k-1)](1+\beta)\mathcal{D}(k)|a_k| \leq (B - A) \qquad (2.1).$$

Proof. A function $f(z)$ of the form (1.2) belongs to the class $\mathcal{T}_q(\lambda, \beta, \vartheta, A, B)$ if and only if there exists an analytic function $w(z)$, satisfying the condition of Schwarz lemma such that

$$(1-\beta)\frac{\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)}{z}+\beta\left(\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)\right)'=\frac{1+Aw(z)}{1+Bw(z)}. \qquad (2.2)$$

or equivalently,

Thus, it is sufficient to show that

$$\left|(1-\beta)\frac{\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)}{z}+\beta\left(\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)\right)'-1\right|-\left|B\left[(1-\beta)\frac{\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)}{z}+\beta\left(\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)\right)\right]-A\right|\leq 0.$$

Letting $|z|=r(0\leq r<1)$, we have

$$\begin{aligned}&\left|(1-\beta)\frac{\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)}{z}+\beta\left(\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)\right)'-1\right|-\left|B\left[(1-\beta)\frac{\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)}{z}+\beta\left(\mathfrak{B}_{\lambda}^{\vartheta,q}f(z)\right)'\right]-A\right|\\&=\left|\sum_{k=2}^{\infty}[1+\beta(k-1)]\mathcal{D}(k)a_kz^{k-1}\right|-\left|(B-A)+B\sum_{k=2}^{\infty}[1+\beta(k-1)]\mathcal{D}(k)a_kz^{k-1}\right|\\&\leq\sum_{k=2}^{\infty}[1+\beta(k-1)]\mathcal{D}(k)|a_k|r^{k-1}-(B-A)+B\sum_{k=2}^{\infty}[1+\beta(k-1)]\mathcal{D}(k)|a_k|r^{k-1}\\&\leq\sum_{k=2}^{\infty}[1+\beta(k-1)](1+B)\mathcal{D}(k)|a_k|-(B-A).\end{aligned}$$

In view of (2.1), the last inequality is less than zero. Hence $f(z)\in\mathcal{T}_q(\lambda,\beta,\vartheta,A,B)$. This completes the proof of Theorem.

Theorem 2.2. Let the function $f(z)\in\mathcal{A}$ be of the form (1.2). Then $f(z)\in\mathcal{T}_q(\lambda,\beta,\vartheta,A,B)$ if and only if

$$\sum_{k=2}^{\infty}[1+\beta(k-1)](1+B)\mathcal{D}(k)|a_k|\leq(B-A). \qquad (2.3)$$

Proof. In view of Theorem 2.1, we need only to show that function $f(z) \in \nu_{\mathrm{q}}(\lambda, \beta, \vartheta, A, B)$ satisfies the coefficient inequalities (2.1). Let $f(z) \in \nu_4(\lambda, \beta, \vartheta, A, B)$.

Then from (1.2) and (2.2), we have

$$\left|\frac{\sum_{k=2}^{\infty} [1+\beta(k-1)]\mathcal{D}(k)z^{k-1}}{(B-A)+\sum_{k=2}^{\infty} B[1+\beta(k-1)]\mathcal{D}(k)a_k z^{k-1}}\right| < 1. \qquad (2.4)$$

Sance $f(z) \in \mathcal{V}, f(z)$ lies in the class $\mathcal{V}(\theta_k, \eta)$ for soene secquence $\{\theta_k\}$ and real number η such that $\theta_k + (k-1)\eta \equiv \pi(\mathrm{mod} 2\pi)$ for all $k \geq 2$.

Set $z = re^{iv}$ in (2.4), we have

which implies

$$\left|\frac{-\sum_{k=2}^{\infty} [1+\beta(k-1)]\mathcal{D}(k)|a_k|r^{k-1}}{(B-A)-B\sum_{k=2}^{\infty} [1+\beta(k-1)]\mathcal{D}(k)|a_k|r^{k-1}}\right| < 1.$$

Since $\Re(w(z)) < |w(z)| < 1$ implies

$$\Re\left[\frac{\sum_{k=2}^{\infty} [1+\beta(k-1)]\mathcal{D}(k)|a_k|r^{k-1}}{(B-A)-B\sum_{k=2}^{\infty} [1+\beta(k-1)]\mathcal{D}(k)|a_k|r^{k-1}}\right] < 1. \qquad (2.5)$$

It has been observed that the denominator of the left hand side of (2.5) cannot vanish for $[0,1)$. Furthermore, it is positive for $r = 0$ and therefore for $r \in [0, 1)$. Thus, we have

$$\sum_{k=2}^{\infty} [1+\beta(k-1)](1+B)\mathcal{D}(k)|a_k|r^{k-1} < (B-A)$$

which, upon letting $r \longrightarrow 1^{-}$gives the require assertion of Theorem .

Corollary 2.1. Let the function $f(z) \in \mathcal{A}$ defined by (1.2) be in the class $\mathcal{T}_q(\lambda, \beta, \vartheta, A, B)$. Then

$$|a_k| \leq \frac{(B-A)}{[1+\beta(\mathrm{k}-1)](1+B)\mathcal{D}(\mathrm{k})}(k \geq 2)$$

The result is sharp for the function

$$f(z) = z + \frac{(B-A)}{[1+\beta(\mathrm{k}-1)](1+B)\mathcal{D}(\mathrm{k})} e^{i\theta_k z^k}(k \geq 2).$$

3 Distortion Theorem

Theorem 3.1. Let the function $f(z)$ defined by (1.1) be in the class $\nu_{\mathrm{q}}(\lambda, \beta, \vartheta, A, B)$. Then

$$|z| - \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}\left|z\right|^2 \leq |f(z)| \leq |z| + \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}\left|z\right|^2 \quad (3.1)$$

The result is sharp.

Proof: Corollary 2.1 and elementary inequality

$$(1+\beta)(1+\mathrm{B})\mathcal{D}(2) \leq [1+\beta(k-1)](1+\mathrm{B})|a_k|\mathcal{D}(k) \leq (B-A). (k \geq 2)$$

yield

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}$$

Thus,

$$|f(z)| = \left|z + \sum_{k=2}^{\infty} a_k z^k\right|$$

$$\leq |z| + \sum_{k=2}^{\infty} |a_k||z|^k$$

$$\leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k|$$

$$\leq |z| + \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)} |z|^2 \qquad (3.2)$$

Similarly, we have

$$|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right|$$

$$\geq |z| - \sum_{k=2}^{\infty} |a_k||z|^k$$

$$\geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k|$$

$$\geq |z| - \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)} |z|^2 \qquad (3.3)$$

Combining (3.2) and (3.3) we obtain the desire result. The result is sharp for the function

$$f(z) = z + \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)} e^{i\theta_2 z^2} \qquad (3.4)$$

at $z = \pm |z| e^{-\mathrm{i}\theta_2}$, This completes the proof.

Theorem 3.2. Let the function $f(z)$ defined by (1.1) belong to the class $\mathcal{V}_q(\lambda, \beta, \vartheta, A, B)$. Then

$$1-\frac{2(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}|z| \leq |f'(z)| \leq 1+\frac{2(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}|z|.$$

The result is sharp for the function $f(z)$ given by (3.4) at $z = \pm|z|e^{-i\theta_2}$.

Proof. In view of the inequality

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}$$

It follows that

$$\sum_{k=2}^{\infty} k|a_k| \leq 2\frac{(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}$$

Thus, we have

$$\begin{aligned} |f'(z)| &= \left|1+\sum_{k=2}^{\infty} ka_k z^{k-1}\right| \\ &\leq 1+|z|\sum_{k=2}^{\infty} k|a_k| \\ &\leq 1+\frac{2(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}|z|. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} |f'(z)| &= \left|1+\sum_{k=2}^{\infty} ka_k z^{k-1}\right| \\ &\geq 1-|z|\sum_{k=2}^{\infty} k|a_k| \\ &\geq 1-\frac{2(B-A)}{(1+\beta)(1+B)\mathcal{D}(2)}|z|. \end{aligned}$$

The result is sharp.

4. Extreme Points

Theorem 4.1. Let the function $f(z)$ defined by (1.1) be in the class $\nu_q(\lambda, \beta, \vartheta, A, B)$ with $\arg(a_k) = \theta_k$where $[\theta_k + (k-1)\eta] \equiv \pi(\mathrm{mod}2\pi)$. Define $f_1(z) = z$ and

$$f_k(z) = z + \frac{(B-A)}{[1+\beta(\mathrm{k}-1)](1+B)\mathcal{D}(\mathrm{k})} e^{i\theta_k} z^k (k \geq 2; z \in E).$$

Then $f(z)$ is in the class $\mathcal{V}_q(\lambda, \beta, \vartheta, A, B)$ if and only if it can be expresesed in the form

$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$,

where $\mu_k \geq 0$ $(\mathrm{k} \geq 0)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof: If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ with $\sum_{k=1}^{\infty} \mu_k = 1$and $\mu_k \geq 0$, then

$$\sum_{k=2}^{\infty} [1+\beta(k-1)](1+B)\mathcal{D}(k) \frac{(B-A)}{[1+\beta(k-1)](1+B)\mathcal{D}(k)} \mu_k$$
$$= \sum_{k=2}^{\infty} (B-A)\mu_k = (B-A)(1-\mu_1) \leq (B-A).$$

So, by Theorem 4.1.15 , we have $f(z) \in \mathcal{V}_q(\lambda, \beta, A, B)$. Conversely, let the function $f(z)$ defined by (1.1) be in the class $\mathcal{V}_q(\lambda, \beta, A, B)$. Define

$$\mu_k = \frac{[1+\beta(k-1)](1+B)\mathcal{D}(k)}{(B-A)} |a_k|, (k \geq 2)$$

and $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$.

We have, $\sum_{k=2}^{\infty} \mu_k \leq 1$ which implies $\mu_1 \geq 0$. Since $\mu_k f_k(z) = \mu_k z + a_k z^k$, we have

$$\sum_{k=1}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} a_k z^k = f(z).$$

This completes the proof.

References

[1] S. Altinkaya and S. Yalcin, Poisson distribution series for certain subclasses of starlike functions with negative coefficients, An. Univ. Oradea Fasc. Mat. 24(2) (2017), 5-8.

[2] S. M. El-Deeb, T. Bulboaca and J. Dziok, Pascal distribution series connected with certain subclasses of univalent functions, Kyungpook Math. J. 59(2) (2019), 301-314.

[3] F. H. Jackson, On q-definite integrals, Quart. J. Pure Appl. Math. 41 (1910), 193-203.

[4] S. Kanas and T. Yaguchi, Subclasses of k-uniformly convex and starlike functions defined by generalized derivative, Publ. Inst. Math. (Beograd) (N.S.) 69(83) (2001), 91-100.

[5] W. Nazeer, Q. Mehmood, S. M. Kang and A. Ul Haq, An application of binomial distribution series on certain analytic functions, J. Comput. Anal. Appl. 26 (2019), 11-17.

[6] S. Porwal and M. Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, Afr. Mat. 27 (2016), 10-21.

[7] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer.*

Math. Soc., **51**, (1975), 109–116.

[8].H. Silverman, Univalent function with varying arguments, Hostan J.Math., 2, (1981),

283-287.

[9] H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. Sci., 44 (2020), 327-344.

[10] A. K. Wanas and J. A. Khuttar, Applications of Borel distribution series on analytic functions, Earthline J. Math. Sci., 4(1) (2020), 71-82.