# Observation on the Sextic equation with three unknowns $(x^2+y^2)-5xy=(k^2+11s^2)z^6$

Dr.A.Kavitha
Professor, Department of Mathematics,
J.J College of Engineering and Technology, Trichy Tamilnadu, India-620 003.

**Abstract:**

The sextic non-homogeneous equation with three unknowns represented by the Diophantine equation $3(x^2+y^2)-5xy=(k^2+11s^2)z^6$ is analysed for its patterns of non-zero distinct integral solutions are illustrated.

**Keywords:**

Integral solutions, Sextic, non-homogeneous equation.

**Introduction:**

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4].. Particularly in [5,6], sextic equations with three unknowns are studied for their integral solutions.. [7] analyse sextic equations with four unknowns for their non-zero integer solutions.. [8] analyse sextic equations with five unknowns for their non-zero solutions. This communication concerns with yet another interesting a non-homogeneous sextic equation with three unknowns given by $(\boldsymbol{x^2+y^2})-\boldsymbol{5xy}=(\boldsymbol{k^2+11s^2})\boldsymbol{z^6}$ **.** Infinitely many non-zero integer tuple (x,y,z) satisfying the above equation are obtained.

**Method of analysis:**

The sextic equation with three unknowns to be solved as

$3(x^2+y^2)-5xy=(k^2+11s^2)z^6$ **,** where k and s are integer**.** **(1)**

The process of obtaining patterns of integral solutions to (1) are illustrated below.

**Pattern: 1**

Introduction of the transformations

$x=u+v,\ \ y=u-v,$ (2)

In (1) leads to

$$u^2 + 11v^2 = (k^2 + 11s^2)z^6 \quad (3)$$

Let $z = a^2 + 11b^2$ (4)

Using (4) in (3) and applying the method of factorization, define

$$u + i\sqrt{11}v = (k + i\sqrt{11}s)(a + i\sqrt{11}b)^6 \quad (5)$$

Equating real and imaginary parts we have,

$$\left.\begin{aligned} u &= \alpha k - 11s\beta \\ v &= \alpha s + \beta k \end{aligned}\right\} \quad (6)$$

Where

$$\left.\begin{aligned} \alpha &= a^6 - 165a^4b^2 + 1815a^2b^4 - 1331b^6 \\ \beta &= 6a^5b - 220a^3b^3 + 726ab^5 \end{aligned}\right\} \quad (7)$$

Using (7) in (2), the corresponding non-zero distinct integral solutions to (1) are given by

$$\left.\begin{aligned} x(a,b) &= \alpha(k+s) + \beta(k-11s) \\ y(a,b) &= \alpha(k-s) - \beta(k+11s) \\ z(a,b) &= a^2 + 11b^2 \end{aligned}\right\} \quad (8)$$

Thus, (8) represents the non-zero distinct integral solution to (1)

**Pattern:2**

Consider (3) as

$$u^2 + 11v^2 = (k^2 + 11s^2)z^6 * 1 \quad (9)$$

Write 1 as

$$1 = \frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100} \quad (10)$$

Substituting (4) and (10) in (9) we get,

$$u + i\sqrt{11}v = \frac{1}{10}\left[(\alpha k - 11\beta s) - 33(\alpha s + \beta k) + i\sqrt{11}\big(3(\alpha k - 11\beta s) + (\alpha s + \beta k)\big)\right]$$

Equating real and imaginary parts we have,

$$\left.\begin{aligned} u &= \frac{1}{10}[(\alpha k - 11\beta s) - 33(\alpha s + \beta k)] \\ v &= \frac{1}{10}[3(\alpha k - 11\beta s) + (\alpha s + \beta k)] \end{aligned}\right\} \quad (11)$$

As our interest is finding integer solutions, we choose α and β suitably so that u and v are integers. Replace a by 10a and b by 10b in (7), substituting the corresponding values of α and β in (11) we get,

$$\left.\begin{aligned} u &= 10^5[(\alpha k - 11\beta s) - 33(\alpha s + \beta k)] \\ v &= 10^5[3(\alpha k - 11\beta s) + (\alpha s + \beta k)] \end{aligned}\right\} \quad (12)$$

Using (12) in (2) we get the non-zero distinct integral solution to (1) is given by

$$\left.\begin{aligned} x(a,b) &= 10^5[4(\alpha k - 11\beta s) - 32(\alpha s + \beta k)] \\ y(a,b) &= 10^5[-2(\alpha k - 11\beta s) - 34(\alpha s + \beta k)] \\ z(a,b) &= 10^1(a^2 + 11b^2) \end{aligned}\right\}$$

**Pattern: 3**

For simplicity and clear understanding we exhibit below the integer solution when k=2 and s=1. For this choice, (1) and (3) simplify respectively to

$$3(x^2 + y^2) - 5xy = 15z^6 \quad (13)$$

$$u^2 + 11v^2 = 15z^6 \quad (14)$$

Write 15 as

$$15 = \left(2 + i\sqrt{11}\right)\left(2 - i\sqrt{11}\right) \quad (15)$$

Using (4) and (15) in (14) and apply the method of factorization, define

$$u + i\sqrt{11}v = \left(2 + i\sqrt{11}\right)\left(\alpha + i\sqrt{11}\beta\right)$$

Equating real and imaginary parts we have,

$$\left.\begin{aligned} u &= 2\alpha - 11\beta \\ v &= \alpha + 2\beta \end{aligned}\right\} \quad (16)$$

Using (16) in (2) we have,

$$\left.\begin{aligned} x(a,b) &= 3\alpha - 9\beta \\ y(a,b) &= \alpha - 13\beta \\ z(a,b) &= a^2 + 11b^2 \end{aligned}\right\} \quad (17)$$

Thus, (17) represent the non-zero distinct integer solution to (13)

Pattern:4

15 can be written as

$$15 = \frac{(7+i\sqrt{11})(7-i\sqrt{11})}{4} \quad (18)$$

Using (4), (18) in (14) and applying the method of factorization define,

$$u + i\sqrt{11}v = \frac{(7+i\sqrt{11})}{2}\left(\alpha + i\sqrt{11}\beta\right)$$

Equating real and imaginary parts, we have,

$$\left.\begin{aligned} u &= \frac{1}{2}(7\alpha - 11\beta) \\ v &= \frac{1}{2}(7\beta + \alpha) \end{aligned}\right\} \quad (19)$$

As our interest is on finding integer solutions, we choose $\alpha$ and $\beta$ suitably so that u and v are integer. Replace a by 2a and b by 2b in (7). Substituting the corresponding values of $\alpha$ and $\beta$ in (19) and employing (2), the non-zero distinct integral solution to (13) are found to be

$$\left.\begin{aligned} x(a,b) &= 2^5(8\alpha - 4\beta) \\ y(a,b) &= 2^5(6\alpha - 17\beta) \\ z(a,b) &= 2^2(a^2 + 11b^2) \end{aligned}\right\}$$

**Conclusion:**

In this paper, we have presented different choices of integer solutions to the sextic equation with three unknowns $3(x^2 + y^2) - 5xy = (k^2 + 11s^2)z^6$ To conclude, as sextic equations are rich in variety, One may consider other forms of sextic equations and their solutions and corresponding properties.

**References**


1. L.E.Dickson, " History of theory of number", Chelsa publishing company, vol-2, New York, 1952
2. L.J.Mordel, "Diophantine equations", Academic Press, New York, (1969).
3. S.J.Telang, " Number Theory", Tata McGraw Hill Publishing company Limited, New Delhi,(2000)
4. D.M.Burton," Elementary Number Theory", Tata McGraw Hill Publishing company Limited, New Delhi, (2002)
5. S.Vidhyalakshmi A.Kavitha M.A.Gopolan, "On the homogeneous sextic equation with three unknowns$3(x^2+y^2)-5xy=36z^6$", Universe of Emerging Technologies and Science, Vol.1, iss. VII, Dec.2014.
6. S.Vidhyalakshmi A.Kavitha M.A.Gopolan, "Integral solution of the sextic equation with three unknowns $(4k-1)(x^2+y^2)-(4k-2)xy=4(4k-1)z^6$", International Journal of Innovation Sciences and Research, Vol.4, pp.323-328, July 2015
7. S.Vidhyalakshmi, M.A.Gopolan and A.Kavitha, "Observations on the non-homogeneous sextic equation with four unknowns$x^3+y^3=2(k^2+3)z^5w$" , International journal of innovative Research in Science, Engineering and Technology, Vol.2, iss.5, May 2013.
8. S.Vidhyalakshmi A.Kavitha M.A.Gopolan, "Observations on the non-homogeneous sextic equation with six unknowns$x^6-y^6-2z^3=(k^2+s^2)^{2n}T^4(w^2-p^2)$", International journal of Advanced in Science, Engineering and Technology, Vol.1,iss.5, Dec.2014.