Some Applications of Erdély-Kober Fractional Derivative Operator

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Abstract

In this article we study a new subclass of analytic functions comprising Erdély-Kober type fractional derivative operator and confer some significant geometric properties like necessary and sufficient condition, growth and distortion bounds, convex combinations,integral means inequality for this newely demarcated class.

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1 Introduction

The theory of analytic function undermines a field that is still actively investigated today despite being an old subject. Many studies on the privileged subject of inequalities in complex analysis have been conducted using the classes of analytical functions. The interaction of geometry and analysis in complex function theory is its most attractive characteristic. These connections between geometric behaviour and analytical structure have been the key area of attention for rapid development. The current work, which developed a new subclass of analytical functions related to the Erdély-Kober Integral Operator, was motivated by this tactic. Several authors have investigated the characteristics of analytic function subclasses and demonstrated how their findings have numerous applications in engineering, hydrodynamics, and signal theory. The extremal difficulties are one of the main issues with geometric function theory. Geometric function theory, the finding of coefficient bounds, sharp estimates, and an extremal function all heavily rely on extremal problems. In the investigation of numerous issues pertaining to the temporal evolution of the free boundary of a viscous fluid for planar flows in Hele-Shaw cells under injection, the theory of analytic univalent functions plays a significant role. The findings we came to in this study could potentially be applicable in other pure and applied disciplines of mathematics.

Let A denote the class of all functions $v(z)$ of the form

$$v(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $v(0) = v'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $v(z)$ which are all univalent in U. A function $v \in A$ is a starlike function of the order $\xi,\ 0 \leq \xi < 1$, if it fulfils

$$\Re\left\{\frac{zv'(z)}{v(z)}\right\} > \xi, \quad z \in U. \tag{1.2}$$

We denote this class with $S^*(\xi)$. A function $u \in A$ is a convex function of the order $\xi,\ \ 0 \leq \xi < 1$, if it fulfils

$$\Re\left\{1 + \frac{zv''(z)}{v'(z)}\right\} > \xi, \quad z \in U. \tag{1.3}$$

We denote this class with $K(\xi)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in U respectively. For $v \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

their convolution (or Hadamard product), denoted by $(v * g)$, is defined as

$$(v * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * v)(z), \quad (z \in U). \tag{1.5}$$

Note that $v * g \in A$.
Let T denotes the class of functions analytic in U that are of the form

$$v(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \ \ (z \in U) \tag{1.6}$$

and let $T^*(\xi) = T \cap S^*(\xi),\ C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [17].
Now we recall the Erdély-Kober type ([8] Ch 5) integral operator definition which shall be used throughout the paper as below:

Definition 1.1. Let for $\vartheta > 0, a, c \in \mathbb{C}$, be such that $\Re(c - a) \geq 0$, an Erdély- Kober type integral operator $\mathscr{I}_{\vartheta}^{a,c} : A \to A$ be defined for $\Re(c - a) > 0$ and $\Re(a) > -\vartheta$ by

$$\mathscr{I}_{\vartheta}^{a,c} v(z) = \frac{\Gamma(c+\vartheta)}{\Gamma(a+\vartheta)} \frac{1}{\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} u(zt^{\vartheta}) dt, \quad \vartheta > 0. \tag{1.7}$$

For $\vartheta > 0, \Re(c - a) \geq 0, \Re(a) > -\vartheta$ and $v \in A$ of the form (1.1) we have

$$\mathscr{I}_{\vartheta}^{a,c} v(z) = z + \sum_{n=2}^{\infty} \mathscr{B}_{\vartheta}^{a,c}(n) a_n z^n. \tag{1.8}$$

where

$$\mathscr{B}_{\vartheta}^{a,c}(n) =) = \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} \quad \text{and} \quad \mathscr{B}_{\vartheta}^{a,c}(2) =) = \frac{\Gamma(c+\vartheta)\Gamma(a+2\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+2\vartheta)} \tag{1.9}$$

Note that $\mathscr{I}_{\vartheta}^{a,a} v(z) = v(z)$.

Definition 1.2. Erdély-Kober fractional order derivative. For $\vartheta > 0, \Re(c-a) \geq 0, \Re(a) > -\vartheta$ $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$; $\ell > -1$; $\varrho > 0$ and $v \in A$ of the form (1.1) we have

$$\mathscr{I}_{\vartheta,m}^{a,c} v(z) = z + \sum_{n=2}^{\infty} \mathscr{B}_{\vartheta,m}^{a,c}(n) a_n z^n. \tag{1.10}$$

where

$$\mathscr{B}^{a,c}_{\vartheta,m}(n) = \left[1 + \frac{\varrho(n-1))}{\ell+1}\right]^m \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} \tag{1.11}$$

Particulary

$$\mathscr{B}^{a,c}_{\vartheta,m}(2) = \left[1 + \frac{\varrho}{\ell+1}\right]^m \frac{\Gamma(c+\vartheta)\Gamma(a+2\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+2\vartheta)}. \tag{1.12}$$

Remark 1.3. By fixing $m = 0$ and suitably choosing the parameters a, c, ϑ as mentioned below, the operator $\mathscr{I}^{a,c}_{\vartheta,m}$ includes various operators studied in the literature as cited below:

(i). For $a = \kappa; c = \varsigma + \kappa$ and $\vartheta = 1$, we obtain the operator $Q^{\varsigma}_{\kappa}v(z)(\varsigma \geq 0; \kappa > 1)$ studied by Jung et al. [7].

(ii). For $a = \varsigma - 1; c = \kappa - 1$ and $\vartheta = 1$, we obtain the operator $\mathscr{L}_{\varsigma,\kappa}v(z)$ $(\varsigma; \kappa \in \mathbb{C} \in \mathbb{Z}_0; \mathbb{Z}_0 = \{0; -1; -2; \cdots\})$ studied by Carlson and Shafer [4].

(iii). For $a = \varsigma - 1; c = \ell$ and $\vartheta = 1$, we obtain the operator $\mathscr{I}_{\varsigma,\ell}$ $(\varsigma > 0; \ell > 0)$ studied by Choi et al [6].

(iv). For $a = \varsigma; c = 0$ and $\vartheta = 1$, we obtain the operator $\mathscr{D}^{\varsigma}$ $(\varsigma > -1)$ studied by Ruschweyh [15].

(v). For $a = 1; c = n$ and $\vartheta = 1$, we obtain the operator $\mathscr{I}_n$ $(n > \mathbb{N}_0$ studied by Noor [13], Noor and Noor [14].

(vi). For $a = \kappa; c = \kappa+1$ and $\vartheta = 1$, we obtain the integral operator $\mathscr{I}_{\kappa,1}$ which studied by Bernardi [3].

(vii). For $a = 1; c = 2$ and $\vartheta = 1$, we obtain the integral operator $\mathscr{I}_{1,1} = I$ which studied by Libera [10] and Livingston [12].

Remark 1.4. By fixing parameters a, c, ϑ as mentioned below, the operator $\mathscr{I}^{a,c}_{\vartheta,m}$ includes various operators studied in the literature as cited below:

(i). By fixing $\ell = 0$ and $\mathscr{I}^{a,a}_{\vartheta,m} \equiv \mathscr{I}^{\varrho}_{\vartheta,m}\left(m \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}\right)$, Al-Oboudi operator [1].

(ii). Assuming $\varrho = 1;\ \ell = 0$ and $\mathscr{I}^{a,a}_{\vartheta,m} \equiv \mathscr{I}^{m}_{\vartheta}\left(m \in \mathbb{N}_0\right)$, Salagean operator [16].

(iii). Assuming $c = 0;\ \vartheta = 1$ and $\mathscr{I}^{a,0}_{1,m} \equiv \mathscr{I}^{a,\ell}_{m,\varrho}, \quad (m \in \mathbb{N}_0)$ studied by Catas in [5].

(iv). By fixing $\varrho = 1;\ \ell = \eta$ and $\mathscr{I}^{a,a}_{\vartheta,-m} \equiv \mathscr{I}^{\eta+1}_{-m}\left(m \in \mathbb{N}_0, \rho \geq 0\right)$ Komatu operator [9].

Now, by making use of the linear operator $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}v(z)$, we define a new subclass of functions belonging to the class A.

Definition 1.5. For $0 \leq \hbar < 1, 0 \leq \sigma < 1,$ *and* $0 < \varsigma < 1$, we let $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar, \sigma, \varsigma)$ be the subclass of v consisting of functions of the form (4) and its geometrical condition satisfy

$$\left| \frac{\hbar\left((\mathscr{I}^{a,c}_{\vartheta,m}v(z))' - \frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}\right)}{\sigma(\mathscr{I}^{a,c}_{\vartheta,m}v(z))' + (1-\hbar)\frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}} \right| < \varsigma, \quad z \in \mathbb{U}$$

where $\mathscr{I}^{a,c}_{\vartheta,m}v(z)$, is given by (1.10).

2 Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$.

Theorem 2.1. *Let the function v be defined by (1.6). Then $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$ if and only if*

$$\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n \leq \varsigma(\sigma+(1-\hbar)), \tag{2.1}$$

where $0<\varsigma<1, 0\leq\hbar<1,$ and $0\leq\sigma<1$. The result (2.1) is sharp for the function

$$v(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}z^n, \quad n\geq 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z|=1$. Then we obtain

$$\begin{aligned}
&\left|\hbar\left((\mathscr{I}^{a,c}_{\vartheta,m}v(z))' - \frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}\right)\right| - \varsigma\left|\sigma\left(\mathscr{I}^{a,c}_{\vartheta,m}v(z))' + (1-\hbar)\frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}\right)\right|\\
=&\left|-\hbar\sum_{n=2}^{\infty}(n-1)\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n z^{n-1}\right|\\
&\quad -\varsigma\left|\sigma+(1-\hbar)-\sum_{n=2}^{\infty}(n\sigma+1-\hbar)\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n z^{n-1}\right|\\
\leq&\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n - \varsigma(\sigma+(1-\hbar))\\
\leq&0.
\end{aligned}$$

Hence, by maximum modulus principle,$v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$. Now assume that $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$ so that

$$\left|\frac{\hbar\left((\mathscr{I}^{a,c}_{\vartheta,m}v(z))' - \frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}\right)}{\sigma(\mathscr{I}^{a,c}_{\vartheta,m}v(z))' + (1-\hbar)\frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}}\right| < \varsigma, \quad z\in\mathbb{U}$$

Hence

$$\left|\hbar\left((\mathscr{I}^{a,c}_{\vartheta,m}v(z))' - \frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}\right)\right| < \varsigma\left|\sigma\left(\mathscr{I}^{a,c}_{\vartheta,m}v(z))' + (1-\hbar)\frac{\mathscr{I}^{a,c}_{\vartheta,m}v(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{n=2}^{\infty}\hbar(n-1)\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n z^{n-1}\right|\\
&< \varsigma\left|\sigma+(1-\hbar)-\sum_{n=2}^{\infty}(n\sigma+1-\hbar)\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n z^{n-1}\right|.
\end{aligned}$$

Thus

$$\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n \leq \varsigma(\sigma+(1-\hbar))$$

and this completes the proof. □

Corollary 2.2. *Let the function $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$.Then*

$$a_n \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}z^n, \quad n\geq 2.$$

3 Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$

Theorem 3.1. *Let the function* $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$. *Then*

$$|z| - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}|z|^2 \le |v(z)| \le |z| + \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}|z|^2.$$

The result is sharp and attained

$$v(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}z^2.$$

Proof.

$$|v(z)| = \left|z - \sum_{n=2}^{\infty} a_n z^n\right| \le |z| + \sum_{n=2}^{\infty} a_n |z|^n \le |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \le \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}. \tag{3.1}$$

Thus

$$|v(z)| \le |z| + \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}|z|^2.$$

Also

$$\begin{aligned} |v(z)| &\ge |z| - \sum_{n=2}^{\infty} a_n |z|^n \\ &\ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\ge |z| - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}|z|^2. \end{aligned}$$

□

Theorem 3.2. *Let* $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$. *Then*

$$1 - \frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}|z| \le |v'(z)| \le 1 + \frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}|z|$$

with equality for

$$v(z) = z - \frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}z^2.$$

Proof. Notice that

$$\begin{aligned}
&[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)\sum_{n=2}^{\infty}na_n\\
\leq&\sum_{n=2}^{\infty}n[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)a_n\\
\leq&\varsigma(\sigma+(1-\hbar)),
\end{aligned}\tag{3.2}$$

from Theorem 2.1. Thus

$$\begin{aligned}
|v'(z)|&=\left|1-\sum_{n=2}^{\infty}na_nz^{n-1}\right|\\
&\leq 1+\sum_{n=2}^{\infty}na_n|z|^{n-1}\\
&\leq 1+|z|\sum_{n=2}^{\infty}na_n\\
&\leq 1+|z|\frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}.
\end{aligned}\tag{3.3}$$

On the other hand

$$\begin{aligned}
|v'(z)|&=\left|1-\sum_{n=2}^{\infty}na_nz^{n-1}\right|\\
&\geq 1-\sum_{n=2}^{\infty}na_n|z|^{n-1}\\
&\geq 1-|z|\sum_{n=2}^{\infty}na_n\\
&\geq 1-|z|\frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(2)}.
\end{aligned}\tag{3.4}$$

Combining (3.3) and (3.4), we get the result. □

4 Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$.

Theorem 4.1. *Let $v\in\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$. Then v is starlike in $|z|<R_1$ of order δ, $0\leq\delta<1$, where*

$$R_1=\inf_n\left\{\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}^{a,c}_{\vartheta,m}(n)}{(n-\delta)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}},\quad n\geq 2.\tag{4.1}$$

Proof. v is starlike of order $\delta, 0\leq\delta<1$ if

$$\Re\left\{\frac{zv'(z)}{v(z)}\right\}>\delta.$$

Thus it is enough to show that

$$\left|\frac{zv'(z)}{v(z)}-1\right|=\left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_nz^{n-1}}\right|\leq\frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zv'(z)}{v(z)}-1\right|\leq 1-\delta \;\; if \;\; \sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)}a_n|z|^{n-1}\leq 1. \tag{4.2}$$

Hence by Theorem 2.1, (4.2) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1}\leq\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z|\leq\left[\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{(n-\delta)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n\geq 2. \tag{4.3}$$

The theorem follows easily from (4.3). □

Theorem 4.2. *Let* $v\in\mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$. *Then* v *is convex in* $|z|<R_2$ *of order* $\delta, 0\leq\delta<1$, *where*

$$R_2=\inf_n\left\{\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{n(n-\delta)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}}, \quad n\geq 2. \tag{4.4}$$

Proof. v is convex of order $\delta, 0\leq\delta<1$ if

$$\Re\left\{1+\frac{zu''(z)}{v'(z)}\right\}>\delta.$$

Thus it is enough to show that

$$\left|\frac{zv''(z)}{v'(z)}\right|=\left|\frac{-\sum\limits_{n=2}^{\infty}n(n-1)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_nz^{n-1}}\right|\leq\frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zv''(z)}{v'(z)}\right|\leq 1-\delta \;\; if \;\; \sum_{n=2}^{\infty}\frac{n(n-\delta)}{(1-\delta)}a_n|z|^{n-1}\leq 1. \tag{4.5}$$

Hence by Theorem 2.1, (4.5) will be true if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1}\leq\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z|\leq\left[\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{n(n-\delta)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n\geq 2. \tag{4.6}$$

The theorem follows easily from (4.6). □

Theorem 4.3. *Let* $v\in\mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$. *Then* v *is close-to-convex in* $|z|<R_3$ *of order* $\delta, \;\; 0\leq\delta<1$, *where*

$$R_3=\inf_n\left\{\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{n\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}}, \quad n\geq 2. \tag{4.7}$$

Proof. v is close-to-convex of order $\delta, 0\leq\delta<1$ if

$$\Re\left\{v'(z)\right\}>\delta.$$

Thus it is enough to show that

$$|v'(z)-1| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus

$$|v'(z)-1| \leq 1-\delta \;\; if \;\; \sum_{n=2}^{\infty} \frac{n}{(1-\delta)} a_n |z|^{n-1} \leq 1. \tag{4.8}$$

Hence by Theorem 2.1, (4.8) will be true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}^{a,c}_{\vartheta,m}(n)}{n\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{4.9}$$

The theorem follows easily from (4.9). □

5 Extreme Points

In the following theorem, we obtain extreme points for the class $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$.

Theorem 5.1. *Let* $v_1(z) = z$ *and*

$$v_n(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)} z^n, \quad for \;\; n = 2,3,\cdots.$$

Then $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$ *if and only if it can be expressed in the form*

$$v(z) = \sum_{n=1}^{\infty} \theta_n v_n(z), \quad where \;\; \theta_n \geq 0 \;\; and \;\; \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $v(z) = \sum\limits_{n=1}^{\infty} \theta_n v_n(z)$, hence we get

$$v(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma(\sigma+(1-\hbar))\theta_n}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)} z^n.$$

Now, $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$, since

$$\begin{aligned}\sum_{n=2}^{\infty} & \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma+(1-\hbar))} \\ & \times \frac{\varsigma(\sigma+(1-\hbar))\theta_n}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)} \\ = & \sum_{n=2}^{\infty} \theta_n = 1-\theta_1 \leq 1.\end{aligned}$$

Conversely, suppose $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$. Then we show that v can be written in the form $\sum\limits_{n=1}^{\infty} \theta_n v_n(z)$.

Now $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar,\sigma,\varsigma)$ implies from Theorem 2.1

$$a_n \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}.$$

Setting $\theta_n = \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma+(1-\hbar))} a_n, n = 2, 3, \cdots$
and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $v(z) = \sum_{n=1}^{\infty} \theta_n v_n(z)$. □

6 Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $\mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar, \sigma, \varsigma)$.

Theorem 6.1. *Let $v, g \in TS(\hbar, \sigma, \varsigma)$. Then $v * g \in TS(\hbar, \sigma, \zeta)$ for*

$$v(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad \textit{and } (v * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2(\sigma + (1-\hbar))\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2 \mathscr{B}^{a,c}_{\vartheta,m}(n) - \varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

Proof. $v \in \mathscr{T}\mathscr{S}^{a,c}_{\vartheta,m}(\hbar, \sigma, \varsigma)$ and so

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma + (1-\hbar))} a_n \leq 1, \tag{6.1}$$

and

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma + (1-\hbar))} b_n \leq 1. \tag{6.2}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\zeta(\sigma + (1-\hbar))} a_n b_n \leq 1. \tag{6.3}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma + (1-\hbar))} \sqrt{a_n b_n} \leq 1. \tag{6.4}$$

Therefore it is enough to show that

$$\frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\zeta(\sigma + (1-\hbar))} a_n b_n \leq \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{\varsigma(\sigma + (1-\hbar))} \sqrt{a_n b_n}.$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\zeta}{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\varsigma}. \tag{6.5}$$

From (6.4)

$$\sqrt{a_n b_n} \leq \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}.$$

Thus it is enough to show that

$$\frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)} \leq \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\zeta}{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma + (1-\hbar))\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2 \mathscr{B}^{a,c}_{\vartheta,m}(n) - \varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

□

7 Closure Theorems

We shall prove the following closure theorems for the class $\mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$.

Theorem 7.1. *Let* $v_j \in \mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma), j = 1, 2, \ldots, s$. *Then*

$$g(z) = \sum_{j=1}^{s} c_j v_j(z) \in \mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$$

For $v_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n$, *where* $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j v_j(z) \\ &= z - \sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n \\ &= z - \sum_{n=2}^{\infty} e_n z^n, \end{aligned}$$

where $e_n = \sum_{j=1}^{s} c_j a_{n,j}$. Thus $g(z) \in \mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$ if

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma + (1-\hbar))} e_n \le 1,$$

that is, if

$$\begin{aligned} &\sum_{n=2}^{\infty}\sum_{j=1}^{s} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma + (1-\hbar))} c_j a_{n,j} \\ =&\sum_{j=1}^{s} c_j \sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma + (1-\hbar))} a_{n,j} \\ \le&\sum_{j=1}^{s} c_j = 1. \end{aligned}$$

□

Theorem 7.2. *Let* $v, g \in \mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$. *Then*

$$h(z) = z - \sum_{n=2}^{\infty}(a_n^2 + b_n^2) z^n \in \mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma), where$$

$$\zeta \ge \frac{2\hbar(n-1)\varsigma^2(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2 \mathscr{B}_{\vartheta,m}^{a,c}(n) - 2\varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

Proof. Since $v, g \in \mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty} \left[\frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma + (1-\hbar))} a_n\right]^2 \le 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma+(1-\hbar))}b_n\right]^2\leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma+(1-\hbar))}\right]^2(a_n^2+b_n^2)\leq 1. \tag{7.1}$$

But $h(z)\in TS(\hbar,\sigma,\zeta)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\zeta(\sigma+(1-\hbar))}(a_n^2+b_n^2)\leq 1, \tag{7.2}$$

where $0<\zeta<1$, however (7.1) implies (7.2) if

$$\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\zeta(\sigma+(1-\hbar))} \leq\frac{1}{2}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\mathscr{B}_{\vartheta,m}^{a,c}(n)}{\varsigma(\sigma+(1-\hbar))}\right]^2.$$

Simplifying, we get

$$\zeta\geq\frac{2\hbar(n-1)\varsigma^2(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\mathscr{B}_{\vartheta,m}^{a,c}(n)-2\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

□

8 Integral Means Inequality

To find the integral means inequality and to verify the Silverman Conjecture [18], $v\in\mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$ we use the following subordination result due to Littlewood [11].

Lemma 8.1. *If $v(z)$ and $u(z)$ are analytic in U with $v(z)\prec u(z)$, then for $\eta>0$, and $z=re^{i\theta}$, $(0<r<1)$,*

$$\int_0^{2\pi}\left|v(re^{i\theta})\right|^\eta d\theta\leq\int_0^{2\pi}\left|u(re^{i\theta})\right|^\eta d\theta.$$

Application of lemma 8.1 to $v(z)$ in the class $\mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$ gives the proof of the following theorem.

Theorem 8.2. *Let $\eta>0$, if $v\in\mathscr{T}\mathscr{S}_{\vartheta,m}^{a,c}(\hbar,\sigma,\varsigma)$, then*

$$\int_0^{2\pi}\left|v(re^{i\theta})\right|^\eta d\theta\leq\int_0^{2\pi}\left|u(re^{i\theta})\right|^\eta d\theta,\quad z=re^{i\theta} and\quad (0<r<1),$$

where $v_2(z)=z-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}z^2$.

Proof. Let $v(z)$ be of the form (1.6) and $v_2(z)=z-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}z^2$. then we show that

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|^\eta d\theta\leq\int_0^{2\pi}\left|1-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}_{\vartheta,m}^{a,c}(n)}\right|^\eta d\theta.$$

By Lemma 8.1, it suffices to show that

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}<1-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}z$$

setting

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}<1-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}w(z) \tag{8.1}$$

using (2.1) and (8.1) we obtain,

$$\begin{aligned}|w(z)&=\left|\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a,c}_{\vartheta,m}(n)}{1-\gamma}a_nz^{n-1}\right|\\&\le|z|\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\mathscr{B}^{a;c}_{\vartheta}(2)}{1-\gamma}|a_n|\\&\le|z|.\end{aligned}$$

□

References

[1] F.M. Al-Oboudi, On univalent functions defined by a generalized salagean operator, Indian J. Math. Math. Sci. 25-28 (2004) 14291436

[2] K.Amarender Reddy and G.Murugusunderamoorthy, A unified class of analytic functions associated with Erdély-Kober integral operator, Int. J. Nonlinear Anal. Appl. (2023), 1 - 13.

[3] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969), 429–446.

[4] B.C. Carlson and D.B. Shafer, Starlike and prestarlike hypergeometric functions, J. Math. Anal. 15 (1984), no4, 737–745.

[5] A. Catas, On a certain differential sandwich theorem associated with a new generalized derivative operator, Gen. Math. 17 (4) (2009) 8395

[6] J.H. Choi, M. Saigo and H.M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276 (2002), 432–445.

[7] I.B. Jung, Y.C. Kim and H.M. Srivastava, The Hardy space of analytic functions associated with certain one parameter families of integral operators, J. Math. Anal. Appl. 176,(1993), 138–147.

[8] V. Kiryakova, Generalized fractional calculus and applications, Pitman Research Notes in Mathematics Series, 301, John Willey and Sons, Inc. New York, 1994.

[9] Y. Komatu, On analytical prolongation of a family of operators, Math. (Cluj) 32 (55) (1990) 141145.

[10] R.J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965), 755-7 58.

[11] J. E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc., 23 (7) (1925), 481–519.

[12] A.E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966),352– 357.

[13] K.I. Noor, On new classes of integral operators, J. Natural Geometry 16 (1999), 71–80.

[14] K.I. Noor and M.A. Noor, On integral operators, J. Math. Anal. Appl. 238 (1999), 341–352.

[15] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109–115.

[16] G.S. Salagean, Subclasses of univalent functions, in: Complex Analysis: Fifth Romanian-Finnish Seminar, PartI (Bucharest, 1981), in: Lecture Notes in Mathematics, vol. 1013, Springer-Verlag, Berlinand New York, 1983.

[17] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51,(1975), 109 - 116.

[18] H. Silverman, Integral means for univalent functions with negative coefficients, Houston Journal of Mathematics, vol. 23, 1997, 169-174.