To Solve Several Definite Integrals by applying the Residue Theorem in the Complex Analysis

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Abstract. Complex analysis is a branch of mathematics that studies the properties and behavior of functions of complex variables, where a complex variable is a quantity that has both a real part andan imaginary part. Complex analysis is important in many areas of science, including physics, engineering, and computer science. The importance of complex analysis lies in its ability to solve problems that are difficult or impossible to solve using only real variables alone. For example, because of the complicated integrals involved, many problems in fluid mechanics, electromagnetism and quantum mechanics can be solved using complex analysis. This paper introduces an importanttheorem in complex analysis, which is the residue theorem. By applying the residue theorem, several types of integrals are transformed into integrals with complex variables, simplifying complexity and difficulty. With the help of examples, the application of the residue theorem is demonstrated. This paper contributes to extending the idea of integral calculation and facilitates the efficient solution of integral calculations in practical problems.

Keywords: Definite integral, Residue's theorem, Derivatives of high order formula, argument principle, Isolated singularities, Non- isolated singularities, Poles and Zeros of holomorphic functions, Analytic function, Complex variable function, essential singular point, single valued function, entire function, simple closed curve Residue Theorem; Singularity; Definite Integral; Complex plane.

1. Introduction

Integrals are a pivotal element of advanced mathematics and an essential part of other interdisciplinary fields. In some practical disciplinary problems, scholars usually need to solve some real integrals. Some simple real integrals can be solved by some regular integration methods by finding the anti derivatives and then applying the Newton-Leibniz formula . However, there are many special forms of integrals whose anti derivatives cannot easily find by regular integral methods. This leads to the fact that scholars cannot use the Newton-Leibniz formula to solve them . If this particular integral cannot be solved, it will cause great difficulties for people conducting research in related fields. This is where an important theory in the field of complex functions is used as a tool forsolving integrals, which is the Residue theorem .

The concept of residues is essential in complex analysis. It is a complex number that represents the coefficient of the term with a negative power in the Laurent series expansion of a complex function . This concept calculates complex integrals with isolated singularities, such as poles or branch points, where the function becomes infinite or undefined. The residue theorem is a useful tool in complex analysis that allows to compute integrals by summing up the residues of the singularities within the contour of integration. By applying the residue theorem, the complex integration can be converted into a simple calculation of the residues, greatly facilitating the overall computational difficulty . Thus, the residue theorem enables to solve integrals that were previously considered impossible to compute by using the regular method. It has pushed the method of solving the value of the definite integral to a new stage, allowing us to calculate integrals that were previously too complexor difficult to evaluate. Moreover, the residue theorem has an essential role in the development and application of complex variable functions, which are used in many branches of mathematics and physics, including electromagnetism, quantum mechanics, and fluid dynamics .

The basic idea of using the residue theorem for integral calculations is: first, to convert a functionof a real variable into an integral of complex variable along a closed loop curve; then, to transform the problem into solving for the residue values at each isolated singular point inside the closed loop;finally, to apply the residue theorem to obtain the solution of the product function. This paper is intended to provide a systematic summary of the residue theorem and to understand the applicationof this important theorem to integral calculations.

2) Method

2.1. Residues and Residue Theorem

Let $f(z)$ be a complex function that is analytic except for a finite number of isolated singularities within a closed and bounded region R. Let C be a simple, closed, and positively oriented contour that encloses all of the singularities of $f(z)$ within R. Then, the residue theorem states that the value of the complex integral $\int_C f(z)dz$ is given by the sum of the residues of $f(z)$ at its singularities within C . A contour integral of a holomorphic function f over a closed curve C is equal to the sum of Residues Resz_k $f(z)$ of the function at all singularities z_k inside the loop multiplies by $2\pi i$:

$$\int_C f(z)dz = 2\pi i \sum_{z_k} Res_{z=z_k} f(z)$$

where the sum is taken over all singularities z_k of $f(z)$ inside C. If z_0 is an isolated singularity of a holomorphic function $f(z)$, often denoted as $Res_{z=z_0} f(z)$, Res_{z_0} or Res(f,z_0). Residue $Res_{z_0} f(z)$is defined as the contour integral around z_0 in a punctured disk divided by $2\pi i$:

$$Res_{z=z_0} f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} f(z)dz$$

2.2. Laurent Series

Let $f(z)$ be a complex function that is holomorphic in an annular region between two concentric circles centered at z_0. That is, $f(z)$ is continuous and differentiable at every point within this region. Suppose the inner circle has radius R_1 and the outer circle has radius R_2, such that $R_1 < |z_0| < R_2$. Then, $f(z)$ can be expressed as a Laurent Series expansion in terms of the variable $(z - z_0)$ as follows:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where the coefficients a_n are given by the line integral:

$$C_n = \frac{1}{2\Pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Here, C is any closed contour that lies entirely within the annular region and encircles the point z_0 in a counterclockwise direction. The integral C denotes the contour integral, which is evaluated by parameterizing the contour and then integrating the resulting expression with respect to the parameter. Furthermore, the residue can be defined by the coefficient for the term $(z - z_0)^{-1}$ of Laurent series: $Res_{z=z_0} f(z) = c_{-1}$.

2.3. Types of Singularities

A singularity of a complex function is a point at which the function fails to be analytic. An isolated singularity of a complex function $f(z)$ is a point z_0 in the complex plane such that $f(z)$ is not holomorphic in any neighborhood of z_0, except possibly at z_0 itself. In other words, $f(z)$ has a singularity at z_0 that is not a limit point of other singularities of $f(z)$. There are three types of isolated singularities: removable singularities, poles, and essential singularities. The first type is removable singularities. A singularity z_0 of a function $f(z)$ is said to be removable if the function can be defined to be analytic at z_0 by assigning a value to $f(z_0)$,

which means $\lim_{z\to z_0} f(z) = c_0$ (c_0 is a complex constant). In other words, the singularity can be "removed" by defining the function appropriately. Removable singularities are characterized by thefact that the function can be extended to be analytic in a neighborhood of z_0. The second type is poles. A singularity z_0 of a function $f(z)$ is said to be a pole of order m if $f^{(n)}(z_0) \neq 0$ $(n = 0,1,2,\dots, m-1)$, $f^{(m)}(z_0) = 0$. The third type is essential singularities. A singularity z_0 of a function $f(z)$ is said to be an essential singularity if it is neither a removable singularity nor a pole. In other words, the function cannot be extended to be analytic in any neighborhood of z_0.

Once the type of isolated singularity on a point has been determined, it is possible to calculate the residue at that point. The residue is equal to 0 for the removable singularities. Besides, the residues on the poles or essential singularities can be defined by the coefficient for the term $(z-z_0)^{-1}$ of Laurent series: $Res_{z\to z_0} f(z) = c_{-1}$. Particularly, the residues for the poles can be defined in a different way: if z_0 is a first order pole, then the residue at z_0 is defined as:

$$Res[f(z), z_0] = \lim_{z\to z_0} (z - z_0) f(z)$$

By contrast, if z_0 is a mth order pole, then the residue at z_0 is defined as

$$Res[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z\to z_0} [(z - z_{0)}{}^m f(z)]^{m-1}$$

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3.Applications of Residue Theorem in Definite Integrals

3.1. Computing $\boldsymbol{R}$(co s θ , si n θ)-type Integral

Let $R(cos\theta, sin\theta)$ be a rational function with real variables. According to Euler's formula $e^{i\theta} = cos\theta + isin\theta$, the integrals of the typical type $\int_0^{2\Pi} R(cos\theta, sin\theta)d\theta$ can be converted by noting $z = e^{i\theta} (0 \le \theta \le 2\pi)$, $cos\theta = \frac{z+z^{-1}}{2}$, $sin\theta = \frac{z-z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$, thus transforming this types of integral into the contour integral of a complex function for the following form:

$$I = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}$$

Then the integral can be evaluated by applying the residue theorem

$$\int_0^{2\Pi} R(cos\theta, sin\theta)d\theta = 2\pi i \sum_c Res[f(z)]$$

This paper will illustrate the method for solving these types of integral through some examples. The first example is

. $$I = \int_0^{2\Pi} \frac{cos2\theta d\theta}{5-4cos\theta}$$

For the z values lying on the unit circle C: {|z| = 1} on the complex plane, they can be defined as $z = e^{i\theta}, (0 \leq \theta \leq 2\pi)$. According to Euler's formula, it can be noted that, $z^2 = cos2\theta + isin2\theta, z^{-2} = cos2\theta - isin2\theta. cos2\theta = \frac{z^2+z^{-2}}{2}, sin2\theta = \frac{z^2-z^{-2}}{2i}, d\theta = \frac{dz}{iz}$.The integrand then be converted into the following form,

$$f(z) = \frac{z^2+z^{-2}}{2iz[5-2(z+z^{-1})]} = \frac{(z^4+1)i}{2z^2(z-2)(2z-1)} \quad .$$

.

The singularities of $f(z)$ are the poles at the points $z_1 = 0$ of the order two and $z_2 = \frac{1}{2}$ and $z_3 = 2$, two simple poles. However, the pointz_3 is not lying inside the unit circle C: {|z|=1}. Thus, it only has the requirement to compute the residues at z_1 and z_2 .

Since $Res[0, f(z)] = \lim_{z\to 0} \frac{d}{dz} z^2 f(z) = \frac{5i}{8}$ and $Res[\frac{1}{2}, f(z)] = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) = -\frac{17i}{24}$, the integral can be solved by applying the Residue theorem, i.e.,

$$I = \int_0^{2\Pi} \frac{cos2\theta d\theta}{5-4cos\theta} = 2\Pi i\left(\frac{5i}{8} - \frac{17i}{24}\right) = \frac{\Pi}{6}$$

The second example is

$$I = \int_0^{2\Pi} cos^{2n}\, \theta d\theta$$

where n is an integer. By Euler's formula, it can be noted that $cos\, \theta = \frac{z+z^{-1}}{2}$, then the above integral can be converted into the form

$$I = \int_c \left(\frac{z+z^{-1}}{2}\right)^{2n} \frac{dz}{iz} = \frac{-i}{z^{2n}} \int_c (\frac{z^2+1}{z^{2n+1}})^{2n}\, dz$$

The integrand $f(z) = (\frac{z^2+1}{z^{2n+1}})^{2n}$ has a singularity at $z = 0$ which has the order of $2n + 1$. Now the computation is available for the residue at this point as

$$Res[0, f(z)] = \lim_{z\to 0}\left[\frac{1}{2n!} \frac{d^{2n}}{dz^{2n}} [(z^2+1)]^{2n}\right]$$

The integral can be computed by applying the residue theorem:

$$I = 2\pi i - \frac{i}{2^{2n}} Res[0, f(z)] = \frac{\pi}{2^{2n-1}} \lim_{z\to 0}\left[\frac{1}{(2n!)} \frac{d^{2n}}{dz^{2n}} [(z^2+1)]^{2n}\right]$$

Thus, the answer can be obtained by expanding the formula to the following form

$$I = \frac{\Pi}{2^{2n-1}(2n)!} \lim_{z\to 0} \frac{d^{2n}}{dz^{2n}} [c_{2n}^0 (z^2)^{2n} + \cdots + c_{2n}^n (z^2)^n] + \cdots + c_{2n}^{2n} = \frac{\Pi}{2^{2n-1}} c_{2n}^n$$

3.2. $\frac{P(x)}{Q(x)}$ type Integral:

The first integral that belongs to this type is

$$I = \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$$

Firstly, since the integrand is an even function,

$$\int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^4} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2} .$$

Then, let the integrand be $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$, which is along the whole real axis on the complex plane. The function here has the isolated singularities at $z = \pm 3i$ which are single poles and $z = \pm 2i$ which are double poles and the function is holomorphic everywhere else. Now set up a semicircle on the complex plane which have the radius greater than 3, the singular points $z = 3i$ and $z = 2i$ of $f(z)$is in the upper half plane in the interior of the semicircular region bounded by the segment $[-r, r]$of the upper half c_r of the circle $|z| = r$ from $-r$ to r and the real axis. Thus, now the residue theorem can be applied by integrating f counterclockwise around the boundary of this semicircular region, the function is now be converted into the form: $\int_{-r}^{r} f(x)dx + \int_{c_r} f(z)dz = 2\pi i[Res(2i, f(z)) + Res(3i, f(z))]$. The residues at the points are

$$Res[2i, f(z)] = \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \frac{z^2}{(z^2+9)(z^2+4)^2} = \lim_{z\to 2i} \frac{d}{dz} (\frac{z^2}{(z^2+9)(z+2i)^2} = \frac{-13i}{200}$$

and

$$Res[3i, f(z)] = \lim_{z\to 3i}(z-3i)\frac{z^2}{(z-3i)(z+3i)(z^2+4)^2} = \frac{3i}{50},$$

thus, the total integral

$$\int_{-r}^{r} f(x)dx + \int_{c_r} f(z)dz = 2\Pi i\left(\frac{-13i}{200} + \frac{3i}{50}\right) = \frac{\Pi}{100}$$

Therefore,

$$\int_{-r}^{r} f(x)dx = \frac{\Pi}{100} - \int_{c_r} f(z)dz$$

which is correct for all the values of r above 3.The value of the integral can be find on the right of the equation approaches 0 as $r \to \infty$. If z is a point on the half circle C_r then $|z^2| = |z|^2 = r^2$
and by the trangle inequality

$|z + w| \geq ||z| - |w||$, then $|(z^2+9)(z^2+4)^2| \geq (||z|^2 - 9|)(||z|^2 - 4^2|) = (r^2-9)(r^2-4)^2$.

Therefore, the required estimate is obtained as follows,

$$|\int_{c_r} f(z)| = |\int_{c_r} \frac{z^2 dz}{(z^2-9)(z^2-4)^2}| \leq \frac{r^2}{(r^2-9)(r^2-4)^2} L(c_r)$$

where $L(C_r) = \pi r$ is the length of the semicircle C_r

. Then it infers that $\left| \int_{C_r} \frac{z^2 dz}{(z^2-9)(z^2-4)^2} \right| \leq \frac{\Pi r^2}{(r^2-9)(r^2-4)^2}$
When $r \to \infty$, the right hand side of the inequality goes to 0, thus, $\int_{C_r} f(z)dz = 0$
.Now, the principal value is P. V. $\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)dx = \frac{\Pi}{100}$. Because the integrand is even, so the Cauchy principal value exists, i.e. $\int_0^{\infty} \frac{x^2 dx}{(x^2-9)(r^2-4)^2} = \frac{\Pi}{200}$.

3.3. $f(x)e^{imx}$ - type Integral:

Here, the first example is

$$\boldsymbol{I = \int_{-\infty}^{\infty} \frac{xe^{2ix}}{x^2-1} dx}$$

By partial fraction, it is observed that the integral has singularities $x = \pm 1$. Hence,

$$I = \lim_{\substack{R\to\infty \\ r_1 r_2 \to 0^+}} \left(\int_{-R}^{-1-r_1} \frac{xe^{2ix}}{x^2-1} dx + \int_{-1+r_1}^{1-r_2} \frac{xe^{2ix}}{x^2-1} dx + \int_{1+r_2}^{R} \frac{xe^{2ix}}{x^2-1} dx \right)$$

Let $I_1 = \int_{S_{r_1}} \frac{ze^{2iz}}{z^2-1} dz$, $I_2 = \int_{S_{r_2}} \frac{ze^{2iz}}{z^2-1} dz$ and $I_R = \int_{C_R} \frac{ze^{2iz}}{z^2-1} dz$. Now $f(z) = \frac{ze^{2iz}}{z^2-1}$

is holomorphic inside the above closed contour. By applying the Cauchy integral theorem,

$$\int_{-R}^{-1-r_1} \frac{xe^{2ix}}{x^2-1} dx + \int_{-1+r_1}^{1-r_2} \frac{xe^{2ix}}{x^2-1} dx + \int_{1+r_2}^{R} \frac{xe^{2ix}}{x^2-1} dx + I_1 + I_2 + I_R = 0$$

By the Jordan Lemma, it follows that when $z \to 0$, $\frac{z}{z^2-1} \to 0$.

So , $\lim_{R\to\infty} I_R = 0$. Since f has simple poles on $z = \pm 1$,

$$\lim_{r_1\to 0^+} I_1 = -i\pi Res(f, -1) = -i\pi \lim_{z\to -1} (z + 1)f(z) = \frac{-i\pi e^{-2i}}{2}$$

Besides, $\lim_{r_2\to 0^+} I_2 = -i\pi Res(f, 1) = \frac{-i\pi e^{2i}}{2}$

$$\text{P. V . } \int_{-\infty}^{\infty} \frac{xe^{2ix}}{x^2-1} dx = \frac{i\pi e^{-2i}}{2} + \frac{i\pi e^{2i}}{2} = i\pi cos(2) \ .$$

The second example is about the Fourier integral

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1}\, dx$$

It is impossible to do the partial fraction for the integrand f, and it can be easy to find that $f(z) = \frac{1}{z^2+z+1}$ has no singularity along the real axis. Besides , $\lim_{z\to\infty} \frac{1}{z^2+z+1} = 0$. However, the integrand does have simple poles on the complex plane which are $z_1 = e^{\frac{2\pi i}{3}}$ and $z_2 = e^{\frac{-2\pi i}{3}}$.

By virtue of the Jordan Lemma, it can be noted that,

$$I = Im \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+x+1}\, dx = Im \oint_C \frac{e^{2iz}}{z^2+z+1}\, dz$$

where C is noted as the union of the infinitely large upper semi-circle and its diameter along the real axis. Thus, it only requires people to compute the residue at z_1.

By applying the residue theorem, the integral can be converted to the form

$$\oint_C \frac{e^{2iz}}{z^2+z+1}\, dz = 2\pi i Res\left(\frac{e^{2iz}}{z^2+z+1}, z_1\right) = 2\pi i \frac{e^{2ie^{\frac{2\pi i}{3}}}}{2e^{\frac{2\pi i}{3}}+1}$$

Therefore, it follows that

$$I = \left(2\pi i \frac{e^{2ie^{\frac{2\pi i}{3}}}}{2e^{\frac{2\pi i}{3}}+1}\right) = \frac{2}{\sqrt{3}}\, \pi e^{-\sqrt{3}} \sin 1$$

The third example is of the form

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx$$

where $a > b > 0$.

Let $p(x) = 1$ and $q(x) = (x^2 + a^2)(x^2 + b^2)$. Clearly, $\deg(q) = 4 > 1 + \deg(p)$.

All the zeros of $q(x)$ are $x = \pm ai, \pm bi$. It can be noted that $q(x) \neq 0$ for all the real x and $x = ai, bi$ are the zeros of $q(x)$ that lie in the upper half plane

Consider the function,

$f(z) = \frac{p(z)}{q(z)} e^{iz} = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$, $(\alpha = 1 > 0)$. The function $f(z)$ has the simple poles at $z = ai$ and $z = bi$ that lie in the upper half plane. Thus, the residues at these points need to be computed.

$$Res[ai, f(z)] = \lim_{z\to ai} (z - ai) \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \lim_{z\to ai} (z + ai) \frac{e^{iz}}{(z+ai)(z^2+b^2)}) = \frac{-ie^{-a}}{2b(b^2-a^2)} ,$$

. . and

$$. \quad Res[bi, f(z)] = \frac{-ie^{-b}}{2b(a^2-b^2)}) .$$

Thus,

$$I = -2\pi\, Im\left(\frac{-ie^{-a}}{2b(b^2-a^2)} + \frac{-ie^{-b}}{2b(a^2-b^2)}\right) = \frac{\pi}{(a^2-b^2)} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

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