A Study on B-K Fixed Point Theorem in Complete-GMS

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Abstract

In the present paper, we established a fixed point theorem in Complete-GMS(Generalized Metric Space), which is a generalization and extension of some of the well known results existing in this literature.

Keywords: Contractive mapping, fixed point, generalized metric space, convergent sequence.

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1.Introduction and Preliminaries

Banach contraction mapping principle is most important and most cited in the literature of fixed point theory. In 1968 Kannan [3] established fixed point result which is also most cited and important in fixed point theory. Later on many autors generalized and extended these results (see for e.g. 1,2 , 4-7) .In 1977 Rhoades [8] analyzed the different types of contractive conditions. In 2000, Branciari [2] introduced generalized metric spaces by replacing triangle inequality by similar ones which involve four or more points instead of three and improved Banach contraction principle. Recently Azam and Arshad [1] obtained results on Kannan fixed point theorem on generalized metric space. In the present paper, we prove a fixed point result on B-K(Banach- Kannan) fixed point theorem in Complete-GMS. Our result is generalization and extension of the results of [1].

For our main results we need some of the following definitions.

Definition 1.1. [1]. Let X be a non- empty set . Suppose the $\rho : X \to X$ satisfies the following

(a) $\rho(x,y) \geq 0$ for all x, y ϵ X and $\rho(x,y) = 0$ if and only if x = y.
(b) $\rho(x,y) = \rho(y, x)$, for all x, y ϵ X.
(c) $\rho(x,y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$, for all x,y ϵ X and for all distinct points w, z ϵ X-{ x, y} [rectangular property].
Then ρ is called a GM-(Generalized Metric) and (X,ρ) is a GMS(Generalized Metric Space).

Definition 1.2. [1]. Let $\{x_n\}$ be a sequence in X and x ϵ X. If for every $\varepsilon > 0$ there is an $n_0 \epsilon N$ such that $\rho(x_n , x) < \varepsilon$,

for all $n > n_0$, then $\{x_n\}$ is said to be convergence , $\{x_n\}$ convergences to x and x is the limit point of $\{x_n\}$. We denote this by $\log_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to\infty$.

Definition 1.3. [1]. Let $\{x_n\}$ be a sequence in X and x ϵ X. If for every $\varepsilon > 0$ there is an $n_0 \epsilon N$ such that

$\rho(x_n , x_{n+m}) < \varepsilon$,for all $n > n_0$, then $\{x_n\}$ is called a Cauchy sequence in X, $\{x_n\}$ convergences to x and x is the limit point of $\{x_n\}$. We denote this by $\log_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to\infty$.

Definition 1.4. [1]. I every Cauchy sequence is convergent in X, then X is called a complete generalized metric space.

Remark 1.1.[1] . (i) $\rho(a_n, y) \to \rho(a, y)$ and $\rho(x, a_n) \to \rho(x, a)$ whenever a_n is a sequence in X with $a_n \to a \epsilon X$.

(ii) X becomes a Hausdroff topological space with neighborhood basis given by :

$B = B\{(x, r): x\epsilon X , r\epsilon(0, \infty) \}$, where $B(x, r) = \{y\epsilon X :\rho(x, y) < r\}$.

3. Main Results

In this section we obtain we obtain our main theorem.

Theorem 3.1. Let (X, ρ) be a CGM-Space and the mapping $A:X\to X$ satisfies the following

$$\rho(Ax, Ay) \leq \alpha_1 \rho(Ax, Ay) + \alpha_2 [\rho(x, Ax) + \rho(y, Ay)]. \quad \ldots \quad (1)$$

For all $x, y \epsilon X$, $\alpha_1, \alpha_2 \epsilon [0,1)$ and $\alpha_1 + 2\alpha_2 < 1$. Then A has a unique fixed point.

Proof: Let x_0 be an arbitrary point in . Let $x_1= A(x_0)$. If $x_1 = x_0$ then $x_0= A(x_0)$ this means that x_0 is a fixed point of A and there is nothing to prove. Assume that $x_1 \neq x_0$, let $x_2= A(x_1)$. In this way we can define a sequence of points in X as follows

$x_{n+1}=Ax_n =A^{n+1}x_0$, $x_n \neq x_{n+1}$, $n =0, 1,2,\ldots$ using (1) we get that

$$\begin{aligned}
\rho(x_n , x_{n+1}) &= \rho(Ax_{n-1} , Ax_n) \\
&\leq \alpha_1 \rho(Ax_{n-1}, Ax_n) + \alpha_2 [\rho(x_{n-1}, Ax_{n-1}) + \rho(x_n, Ax_n)], \\
&\leq \alpha_1 \rho(x_{n-1}, x_n) + \alpha_2 [\rho(x_{n-1}, x_n) + \rho(x_n, x_{n+1})], \\
&\leq [\alpha_1 + \alpha_2) / 1- \alpha_2] \ \rho(x_{n-1}, x_n) , \\
&\leq \lambda \rho(x_{n-1}, x_n). \quad \ldots \quad (2)
\end{aligned}$$

Where , $\lambda = [\alpha_1 + \alpha_2) / 1- \alpha_2] < 1$.

Now we can also suppose that x_0 is not a periodic point. In fact if $x_n = x_0$, then

$\rho(x_0 , Ax_0) = \rho(x_n , Ax_n)$

$\leq \rho(A^n x_0, A^n x_0),$

$\leq \lambda \rho(A^{n-1} x_0, A^n x_0) \leq \lambda^2 \rho(A^{n-2} x_0, A^{n-1} x_0) \leq \ldots \leq \lambda^n \rho(x_0, A x_0)$

$\rho(x_0 , Ax_0) \leq \lambda^n \rho(x_0, A x_0),$

$(1-\lambda^n) \rho(x_0, A x_0) \leq 0.$

It follows that x_0 is a fixed point of A. Thus in the sequel of point we can suppose that $A^n x_0 \neq x_0$, for n = 1,2,3,…

Now (1) implies that

$\rho(A^n x_0, A^{n+m} x_0) = (\alpha_1 + \alpha_2)\rho(A^{n-1} x_0, A^n x_0) + \alpha_2 \rho(A^{n+m-1} x_0, A^{n+m} x_0),$

$\leq (\alpha_1 + \alpha_2) \lambda^{n-1}\rho(x_0, A x_0) + \lambda^{n+m-1} \rho(x_0, Ax_0).$

Therefore, $\rho(x_n, x_{n+m}) \to 0$ as $m, n \to\infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists a $p\epsilon X$ such that $x_n \to p$. By reciprocal property we get that

$\rho(Ap, p) \leq \rho(Ap, A^n x_0) + \rho(A^n x_0, A^{n+1} x_0) + \rho(A^{n+1} x_0, p),$

$\leq \alpha_1 \rho(p, A^{n-1} x_0) + \alpha_2 [\rho(p, Ap) + \rho(A^{n-1} x_0 , A^n x_0)] + \lambda^n\rho(p, A^{n-1} x_0) + \rho(A^{n+1} x_0 , p),$

$\leq (\alpha_1 + \alpha_2) / (1- \alpha_2) \rho(A^{n-1} x_0 , A^n x_0) + (\lambda^n /1-\alpha_2)\rho(x_0 , A x_0) + (1/ 1-\alpha_2)\rho(A^{n+1} x_0 , p)$.

Letting $n \to \infty$ we get that

Using the fact that $\rho(a_n, y) \to \rho(a, y)$ and $\rho(x, a_n) \to \rho(x, a)$ whenever $\{a_n\}$ is a sequence in X with $a_n \to a \in X$,

We get that $p = Ap$. Therefore A has a fixed point. Now we prove that A has a unique fixed point.. Now we assume that there exists another point $q \in X$ such that $q = Aq$.

Now $\rho(q, p) = \rho(Aq, Ap)$

$$\leq \alpha_1 \rho(q, p) + \alpha_2 [\rho(q, Aq) + \rho(p, Ap)],$$

$$\leq \alpha_1 \rho(q, p) + \alpha_2 [\rho(q, qq) + \rho(p, p)],$$

$\rho(q, p) \leq 0$. Implies that $\rho(q, p) = 0$, that is $p = q$.

Remark 2.2. If we take $\alpha_1 = 0$ and $\alpha_2 = \lambda$ in the above theorem 2.1 we can get the theorem 2.1 of [1].

Conclusion: In this paper our results are generalized results and are more general than the results of [1].

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